scope of quantifier

Parentheses, as you can see from this example, make a big difference. They are the way you can tell what the *scope* of a quantifier is, that is, which variables fall under its influence and which don't. This example also shows that a variable can occur both free and bound in a formula. It is really an *occurrence* of a variable that is either free or bound, not the variable itself. In the formula

$$\exists x\, \mathsf{Doctor}(x) \land \mathsf{Smart}(x)$$

the last occurrence of x is free and the second is bound.

Remember

1. Complex wffs are built from atomic wffs by means of truth-functional connectives and quantifiers in accord with the rules on page 233.
2. When you append either quantifier $\forall x$ or $\exists x$ to a wff P, we say that the quantifier binds all the free occurrences of x in P.
3. A sentence is a wff in which no variables occur free (unbound).

Exercises

9.1 (Fixing some expressions) Open the sentence file Bernstein's Sentences. The expressions in this list are not quite well-formed sentences of our language, but they can all be made sentences by slight modification. Turn them into sentences *without adding or deleting any quantifier symbols*. With some of them, there is more than one way to make them a sentence. Use **Verify** to make sure your results are sentences and then submit the corrected file.

9.2 (Fixing some more expressions) Open the sentence file Schonfinkel's Sentences. Again, the expressions in this list are not well-formed sentences. Turn them into sentences, but this time, do it *only* by adding quantifier symbols or variables, or both. Do not add any parentheses. Use **Verify** to make sure your results are sentences and submit the corrected file.

9.3 (Making them true) Open Bozo's Sentences and Leibniz's World. Some of the expressions in this file are not wffs, some are wffs but not sentences, and one is a sentence but false. Read and assess each one. See if you can adjust each one to make it a true sentence with as little change as possible. Try to capture the intent of the original expression, if you can tell what that was (if not, don't worry). Use **Verify** to make sure your results are true sentences and then submit your file.

2. Whether you evaluated the sentence correctly or not, play the game twice for each sentence, first committed to TRUE, then committed to FALSE. Make sure you understand how the game works at each step. ◄

3. There is nothing to save except your understanding of the game. ◄

. . . . *Congratulations*

Exercises

9.4 ✎ If you skipped the **You try it** section, go back and do it now. This is an easy but important exercise that will familiarize you with the game rules for the quantifiers. There is nothing you need to turn in or submit.

9.5 ➶ (Evaluating sentences in a world) Open Peirce's World and Peirce's Sentences. There are 30 sentences in this file. Work through them, assessing their truth and playing the game when necessary. Make sure you understand why they have the truth values they do. (You may need to switch to the 2-D view for some of the sentences.) After you understand each of the sentences, go back and make the false ones true by adding or deleting a negation sign. Submit the file when the sentences all come out true in Peirce's World.

9.6 ➶ (Evaluating sentences in a world) Open Leibniz's World and Zorn's Sentences. The sentences in this file contain both quantifiers and the identity symbol. Work through them, assessing their truth and playing the game when necessary. After you're sure you understand why the sentences get the values they do, modify the false ones to make them true. But this time you can make any change you want *except* adding or deleting a negation sign.

9.7 ✎ In English we sometimes say things like *Every Jason is envied*, meaning that everyone named "Jason" is envied. For this reason, students are sometimes tempted to write expressions like ∀b Cube(b) to mean something like *Everything named **b** is a cube.* Explain why this is not well formed according to the grammatical rules on page 233.

Section 9.5

The four Aristotelian forms

Long before FOL was codified, Aristotle studied the kinds of reasoning associated with quantified noun phrases like *Every man*, *No man*, and *Some man*, expressions we would translate using our quantifier symbols. The four main sentence forms treated in Aristotle's logic were the following.

Aristotelian forms

All P's are Q's
Some P's are Q's
No P's are Q's
Some P's are not Q's

We will begin by looking at the first two of these forms, which we have already discussed to a certain extent. These forms are translated as follows. The form *All P's are Q's* is translated as:

$$\forall x\,(P(x) \rightarrow Q(x))$$

whereas the form *Some P's are Q's* is translated as:

$$\exists x\,(P(x) \land Q(x))$$

Beginning students are often tempted to translate the latter more like the former, namely as:

$$\exists x\,(P(x) \rightarrow Q(x))$$

This is in fact an extremely unnatural sentence of first-order logic. It is meaningful, but it doesn't mean what you might think. It is true just in case there is an object which is either not a P or else is a Q, which is something quite different than saying that some P's are Q's. We can quickly illustrate this difference with Tarski's World.

You try it

▶ 1. Use Tarski's World to build a world containing a single large cube and nothing else.

▶ 2. Write the sentence $\exists x\,(\mathsf{Cube}(x) \rightarrow \mathsf{Large}(x))$ in the sentence window. Check to see that the sentence is true in your world.

▶ 3. Now change the large cube into a small tetrahedron and check to see if the sentence is true or false. Do you understand why the sentence is still true? Even if you do, play the game twice, once committed to its being false, once to its being true.

▶ 4. Add a second sentence that correctly expresses the claim that there is a large cube. Make sure it is false in the current world but becomes true when you add a large cube. Save your two sentences as Sentences Quantifier 1.

. . . . ***Congratulations***

The other two Aristotelian forms are translated similarly, but using a negation. In particular *No P's are Q's* is translated

$$\forall x\, (P(x) \rightarrow \neg Q(x))$$

Many students, and one of the authors, finds it more natural to use the following, logically equivalent sentence:

$$\neg \exists x\, (P(x) \wedge Q(x))$$

Both of these assert that nothing that is a P is also a Q.

The last of the four forms, *Some P's are not Q's*, is translated by

$$\exists x\, (P(x) \wedge \neg Q(x))$$

which says there is something that is a P but not a Q.

The four Aristotelian forms are the very simplest sorts of sentences built using quantifiers. Since many of the more complicated forms we talk about later are elaborations of these, you should learn them well.

Remember

The four Aristotelian forms are translated as follows:

All P's are Q's.	$\forall x\, (P(x) \rightarrow Q(x))$
Some P's are Q's.	$\exists x\, (P(x) \wedge Q(x))$
No P's are Q's.	$\forall x\, (P(x) \rightarrow \neg Q(x))$
Some P's are not Q's.	$\exists x\, (P(x) \wedge \neg Q(x))$

Exercises

9.8 If you skipped the **You try it** section, go back and do it now. Submit the file Sentences Quantifier 1.

9.9 (Building a world) Open Aristotle's Sentences. Each of these sentences is of one of the four Aristotelian forms. Build a single world where all the sentences in the file are true. As you work through the sentences, you will find yourself successively modifying the world. Whenever you make a change in the world, you had better go back and check that you haven't made any of the earlier sentences false. Then, when you are finished, verify that all the sentences are really true and submit your world.

9.10 ✎ (Common translation mistakes) Open Edgar's Sentences and evaluate them in Edgar's World. Make sure you understand why each of them has the truth value it does. Play the game if any of the evaluations surprise you. Which of these sentences would be a good translation of *There is a tetrahedron that is large?* (Clearly this English sentence is false in Edgar's World, since there are no tetrahedra at all.) Which sentence would be a good translation of *There is a cube between **a** and **b**?* Which would be a good translation of *There is a large dodecahedron?* Express in clear English the claim made by each sentence in the file and turn in your answers to your instructor.

9.11 ➶|✎ (Common mistakes, part 2) Open Allan's Sentences. In this file, sentences 1 and 4 are the correct translations of *Some dodecahedron is large* and *All tetrahedra are small*, respectively. Let's investigate the logical relations between these and sentences 2 and 3.

1. Construct a world in which sentences 2 and 4 are true, but sentences 1 and 3 are false. Save it as World 9.11.1. This shows that sentence 1 is not a consequence of 2, and sentence 3 is not a consequence of 4.
2. Can you construct a world in which sentence 3 is true and sentence 4 is false? If so, do so and save it as World 9.11.2. If not, explain why you can't and what this shows.
3. Can you construct a world in which sentence 1 is true and sentence 2 is false? If so, do so and save it as World 9.11.3. If not, explain why not.

Submit any world files you constructed and turn in any explanations to your instructor.

9.12 ➶ (Describing a world) Open Reichenbach's World 1. Start a new sentence file where you will describe some features of this world using sentences of the simple Aristotelian forms. Check each of your sentences to see that it is indeed a sentence and that it is true in this world.

1. Use your first sentence to describe the size of all the tetrahedra.
2. Use your second sentence to describe the size of all the cubes.
3. Use your third sentence to express the truism that every dodecahedron is either small, medium, or large.
4. Notice that some dodecahedron is large. Express this fact.
5. Observe that some dodecahedron is not large. Express this.
6. Notice that some dodecahedron is small. Express this fact.
7. Observe that some dodecahedron is not small. Express this.
8. Notice that some dodecahedron is neither large nor small. Express this.
9. Express the observation that no tetrahedron is large.
10. Express the fact that no cube is large.

Now change the sizes of the objects in the following way: make one of the cubes large, one of the tetrahedra medium, and all the dodecahedra small. With these changes, the following should come out false: 1, 2, 4, 7, 8, and 10. If not, then you have made an error in describing the original world. Can you figure out what it is? Try making other changes and see if your sentences have the expected truth values. Submit your sentence file.

9.13 ✎ Assume we are working in an extension of the first-order language of arithmetic with the additional predicates Even(x) and Prime(x), meaning, respectively, "x is an even number" and "x is a prime number." Create a sentence file in which you express the following claims:

1. Every even number is prime.
2. No even number is prime.
3. Some prime is even.
4. Some prime is not even.
5. Every prime is either odd or equal to 2.

[Note that you should assume your domain of discourse consists of the natural numbers, so there is no need for a predicate Number(x). Also, remember that 2 is not a constant in the language, so must be expressed using + and 1.]

9.14 ✎⋆ (Name that object) Open Maigret's World and Maigret's Sentences. The goal is to try to figure out which objects have names, and what they are. You should be able to figure this out from the sentences, all of which are true. Once you have come to your conclusion, assign the six names to objects in the world in such a way that all the sentences do indeed evaluate as true. Submit your modified world.

Section 9.6

Translating complex noun phrases

The first thing you have to learn in order to translate quantified English expressions is how to treat complex noun phrases, expressions like "a boy living in Omaha" or "every girl living in Duluth." In this section we will learn how to do this. We concentrate first on the former sort of noun phrase, whose most natural translation involves an existential quantifier. Typically, these will be noun phrases starting with one of the determiners *some*, *a*, and *an*, including noun phrases like *something*. These are called existential noun phrases, since they assert the existence of something or other. Of course two of our four Aristotelian forms involve existential noun phrases, so we know the general pattern: existential noun phrases are usually translated using ∃, frequently together with ∧.

existential noun phrases

Let's look at a simple example. Suppose we wanted to translate the sentence *A small, happy dog is at home*. This sentence claims that there is an object which is simultaneously a small, happy dog, and at home. We would translate it as

$$\exists x\,[(\mathsf{Small}(x) \land \mathsf{Happy}(x) \land \mathsf{Dog}(x)) \land \mathsf{Home}(x)]$$

no freshman took her class. Here we wouldn't say that she lied, but we would certainly say that she misled us. Her statement typically carries the conversational implicature that there were freshmen in the class. If there were no freshmen, then that's what she would have said if she were being forthright. *Inherently* vacuous claims are true only when they are misleading, so they strike us as intuitively false.

conversational implicature

Another source of confusion concerns the relationship between the following two Aristotelian sentences:

Some P's are Q's
All P's are Q's

Students often have the intuition that the first should contradict the second. After all, why would you say that *some* student got an A if *every* student got an A? If this intuition were right, then the correct translation of *Some P's are Q's* would not be what we have suggested above, but rather

$$\exists x\,(P(x) \land Q(x)) \land \neg\forall x\,(P(x) \to Q(x))$$

It is easy to see, however, that the second conjunct of this sentence does not represent part of the meaning of the sentence. It is, rather, another example of a conversational implicature. It makes perfectly good sense to say "Some student got an A on the exam. In fact, every student did." If the proposed conjunction were the right form of translation, this amplification would be contradictory.

Remember

1. *All P's are Q's* does not imply, though it may conversationally suggest, that there are some P's.
2. *Some P's are Q's* does not imply, though it may conversationally suggest, that not all P's are Q's.

Exercises

9.15 ➶ If you skipped the **You try it** section, go back and do it now. Submit the file Sentences Vacuous 1.

9.16 (Translating existential noun phrases) Start a new sentence file and enter translations of the following English sentences. Each will use the symbol ∃ exactly once. None will use the symbol ∀. As you go, check that your entries are well-formed sentences. By the way, you will find that many of these English sentences are translated using the same first-order sentence.

1. *Something is large.*
2. *Something is a cube.*
3. *Something is a large cube.*
4. *Some cube is large.*
5. *Some large cube is to the left of **b**.*
6. *A large cube is to the left of **b**.*
7. ***b** has a large cube to its left.*
8. ***b** is to the right of a large cube.* [Hint: This translation should be almost the same as the last, but it should contain the predicate symbol RightOf.]
9. *Something to the left of **b** is in back of **c**.*
10. *A large cube to the left of **b** is in back of **c**.*
11. *Some large cube is to the left of **b** and in back of **c**.*
12. *Some dodecahedron is not large.*
13. *Something is not a large dodecahedron.*
14. *It's not the case that something is a large dodecahedron.*
15. ***b** is not to the left of a cube.* [Warning: This sentence is ambiguous. Can you think of two importantly different translations? One starts with ∃, the other starts with ¬. Use the second of these for your translation, since this is the most natural reading of the English sentence.]

Now let's check the translations against a world. Open Montague's World.

- Notice that all the English sentences above are true in this world. Check that all your translations are also true. If not, you have made a mistake. Can you figure out what is wrong with your translation?
- Move the large cube to the back right corner of the grid. Observe that English sentences 5, 6, 7, 8, 10, 11, and 15 are now false, while the rest remain true. Check that the same holds of your translations. If not, you have made a mistake. Figure out what is wrong with your translation and fix it.
- Now make the large cube small. The English sentences 1, 3, 4, 5, 6, 7, 8, 10, 11, and 15 are false in the modified world, the rest are true. Again, check that your translations have the same truth values. If not, figure out what is wrong.
- Finally, move *c* straight back to the back row, and make the dodecahedron large. All the English sentences other than 1, 2, and 13 are false. Check that the same holds for your translations. If not, figure out where you have gone wrong and fix them.

When you are satisfied that your translations are correct, submit your sentence file.

9.17 (Translating universal noun phrases) Start a new sentence file, and enter translations of the following sentences. This time each translation will contain exactly one ∀ and no ∃.

1. *All cubes are small.*
2. *Each small cube is to the right of **a**.*
3. ***a** is to the left of every dodecahedron.*
4. *Every medium tetrahedron is in front of **b**.*
5. *Each cube is either in front of **b** or in back of **a**.*
6. *Every cube is to the right of **a** and to the left of **b**.*
7. *Everything between **a** and **b** is a cube.*
8. *Everything smaller than **a** is a cube.*
9. *All dodecahedra are not small.* [Note: Most people find this sentence ambiguous. Can you find both readings? One starts with ∀, the other with ¬. Use the former, the one that means all the dodecahedra are either medium or large.]
10. *No dodecahedron is small.*
11. ***a** does not adjoin everything.* [Note: This sentence is ambiguous. We want you to interpret it as a denial of the claim that *a* adjoins everything.]
12. ***a** does not adjoin anything.* [Note: These last two sentences mean different things, though they can both be translated using ∀, ¬, and Adjoins.]
13. ***a** is not to the right of any cube.*
14. (⋆) *If something is a cube, then it is not in the same column as either **a** or **b**.* [Warning: While this sentence contains the noun phrase "something," it is actually making a universal claim, and so should be translated with ∀. You might first try to paraphrase it using the English phrase "every cube."]
15. (⋆) *Something is a cube if and only if it is not in the same column as either **a** or **b**.*

Now let's check the translations in some worlds.

- Open Claire's World. Check to see that all the English sentences are true in this world, then make sure the same holds of your translations. If you have made any mistakes, fix them.

- Adjust Claire's World by moving *a* directly in front of *c*. With this change, the English sentences 2, 6, and 12–15 are false, while the rest are true. Make sure that the same holds of your translations. If not, try to figure out what is wrong and fix it.

- Next, open Wittgenstein's World. Observe that the English sentences 2, 3, 7, 8, 11, 12, and 13 are true, but the rest are false. Check that the same holds for your translations. If not, try to fix them.

- Finally, open Venn's World. English sentences 2, 4, 7, and 11–14 are true; does the same hold for your translations?

When you are satisfied that your translations are correct, submit your sentence file.

9.18 (Translation) Open Leibniz's World. This time, we will translate some sentences while looking at the world they are meant to describe.

- Start a new sentence file, and enter translations of the following sentences. Each of the English sentences is true in this world. As you go, check to make sure that your translation is indeed a true sentence.

  1. *There are no medium-sized cubes.*
  2. *Nothing is in front of **b**.*
  3. *Every cube is either in front of or in back of **e**.*
  4. *No cube is between **a** and **c**.*
  5. *Everything is in the same column as **a**, **b**, or **c**.*

- Now let's change the world so that none of the English sentences is true. We can do this as follows. First change *b* into a medium cube. Next, delete the leftmost tetrahedron and move *b* to exactly the position just vacated by the late tetrahedron. Finally, add a small cube to the world, locating it exactly where *b* used to sit. If your answers to 1–5 are correct, all of the translations should now be false. Verify that they are.

- Make various changes to the world, so that some of the English sentences come out true and some come out false. Then check to see that the truth values of your translations track the truth values of the English sentences.

9.19 ⋆ Start a new sentence file and translate the following into FOL using the symbols from Table 1.2, page 30. Note that all of your translations will involve quantifiers, though this may not be obvious from the English sentences. (Some of your translations will also require the identity predicate.)

1. *People are not pets.*
2. *Pets are not people.*
3. *Scruffy was not fed at either 2:00 or 2:05.* [Remember, Fed is a ternary predicate.]
4. *Claire fed Folly at some time between 2:00 and 3:00.*
5. *Claire gave a pet to Max at 2:00.*
6. *Claire had only hungry pets at 2:00.*
7. *Of all the students, only Claire was angry at 3:00.*
8. *No one fed Folly at 2:00.*
9. *If someone fed Pris at 2:00, they were angry.*
10. *Whoever owned Pris at 2:00 was angry five minutes later.*

9.20 ✎⋆ Using Table 1.2, page 30, translate the following into colloquial English.

1. $\forall t\, \neg \mathsf{Gave}(\mathsf{claire}, \mathsf{folly}, \mathsf{max}, \mathsf{t})$
2. $\forall x\, (\mathsf{Pet}(x) \rightarrow \mathsf{Hungry}(x, 2{:}00))$

3. $\forall y\,(\mathsf{Person}(y) \rightarrow \neg\mathsf{Owned}(y, \mathsf{pris}, 2{:}00))$
4. $\neg\exists x\,(\mathsf{Angry}(x, 2{:}00) \wedge \mathsf{Student}(x) \wedge \mathsf{Fed}(x, \mathsf{carl}, 2{:}00))$
5. $\forall x\,((\mathsf{Pet}(x) \wedge \mathsf{Owned}(\mathsf{max}, x, 2{:}00)) \rightarrow \mathsf{Gave}(\mathsf{max}, x, \mathsf{claire}, 2{:}00))$

9.21 ✎⋆⋆ Translate the following into FOL, introducing names, predicates, and function symbols as needed. As usual, explain your predicates and function symbols, and any shortcomings in your translations. If you assume a particular domain of discourse, mention that as well.

1. *Only the brave know how to forgive.*
2. *No man is an island.*
3. *I care for nobody, not I,*
   If no one cares for me.
4. *Every nation has the government it deserves.*
5. *There are no certainties, save logic.*
6. *Misery (that is, a miserable person) loves company.*
7. *All that glitters is not gold.*
8. *There was a jolly miller once*
   Lived on the River Dee.
9. *If you praise everybody, you praise nobody.*
10. *Something is rotten in the state of Denmark.*

Section 9.7

Quantifiers and function symbols

When we first introduced function symbols in Chapter 1, we presented them as a way to form complex names from other names. Thus $\mathsf{father}(\mathsf{father}(\mathsf{max}))$ refers to Max's father's father, and $(1 + (1 + 1))$ refers to the number 3. Now that we have variables and quantifiers, function symbols become much more useful than they were before. For example, they allow us to express in a very compact way things like:

$$\forall x\ \mathsf{Nicer}(\mathsf{father}(\mathsf{father}(x)), \mathsf{father}(x))$$

This sentence says that everyone's paternal grandfather is nicer than their father, a false belief held by many children.

Notice that even if our language had individual constants naming everyone's father (and their fathers' fathers and so on), we could not express the above claim in a single sentence without using the function symbol father. True, if we added the binary predicate FatherOf, we could get the same point across, but the sentence would be considerably more complex. It would require three universal quantifiers, something we haven't talked about yet:

$$\forall x\, \forall y\, \forall z\, ((\mathsf{FatherOf}(x, y) \land \mathsf{FatherOf}(y, z)) \to \mathsf{Nicer}(x, y))$$

In our informal mathematical examples, we have in fact been using function symbols along with variables throughout the book. For example in Chapter 8, we proved the conditional:

$$\mathsf{Even}(n^2) \to \mathsf{Even}(n)$$

This sentence is only partly in our official language of first-order arithmetic. Had we had quantifiers at the time, we could have expressed the intended claim using a universal quantifier and the binary function symbol $\times$:

$$\forall y\, (\mathsf{Even}(y \times y) \to \mathsf{Even}(y))$$

The blocks language does not have function symbols, though we could have introduced some. Remember the four function symbols, fm, bm, lm and rm, that we discussed in Chapter 1 (page 33). The idea was that these meant *frontmost, backmost, leftmost,* and *rightmost*, respectively, where, for instance, the complex term lm(b) referred to the leftmost block in the same row as *b*. Thus a formula like

$$\mathsf{lm}(x) = x$$

is satisfied by a block *b* if and only if *b* is the leftmost block in its row. If we append a universal quantifier to this atomic wff, we get the sentence

$$\forall x\, (\mathsf{lm}(x) = x)$$

which is true in exactly those worlds that have at most one block in each row. This claim could be expressed in the blocks language without function symbols, but again it would require a sentence with more than one quantifier.

To check if you understand these function symbols, see if you can tell which of the following two sentences is true in all worlds and which makes a substantive claim, true in some worlds and false in others:

$$\forall x\, (\mathsf{lm}(\mathsf{lm}(x)) = \mathsf{lm}(x))$$
$$\forall x\, (\mathsf{fm}(\mathsf{lm}(x)) = \mathsf{lm}(x))$$

In reading a term like fm(lm(b)), remember that you apply the *inner* function first, then the outer. That is, you first find the leftmost block in the row containing *b*—call it *c*—and then find the frontmost block in the column containing *c*.

Function symbols are extremely useful and important in applications of FOL. We close this chapter with some problems that use function symbols.

Exercises

9.22 ✎ Assume that we have expanded the blocks language to include the function symbols fm, bm, lm and rm described earlier. Then the following formulas would all be sentences of the language:

1. $\exists y\,(\mathsf{fm}(y) = e)$
2. $\exists x\,(\mathsf{lm}(x) = b \land x \neq b)$
3. $\forall x\, \mathsf{Small}(\mathsf{fm}(x))$
4. $\forall x\,(\mathsf{Small}(x) \leftrightarrow \mathsf{fm}(x) = x)$
5. $\forall x\,(\mathsf{Cube}(x) \rightarrow \mathsf{Dodec}(\mathsf{lm}(x)))$
6. $\forall x\,(\mathsf{rm}(\mathsf{lm}(x)) = x)$
7. $\forall x\,(\mathsf{fm}(\mathsf{bm}(x)) = x)$
8. $\forall x\,(\mathsf{fm}(x) \neq x \rightarrow \mathsf{Tet}(\mathsf{fm}(x)))$
9. $\forall x\,(\mathsf{lm}(x) = b \rightarrow \mathsf{SameRow}(x, b))$
10. $\exists y\,(\mathsf{lm}(\mathsf{fm}(y)) = \mathsf{fm}(\mathsf{lm}(y)) \land \neg \mathsf{Small}(y))$

Fill in the following table with TRUE's and FALSE's according to whether the indicated sentence is true or false in the indicated world. Since Tarski's World does not understand the function symbols, you will not be able to check your answers. We have filled in a few of the entries for you. Turn in the completed table to your instructor.

	Malcev's	Bolzano's	Boole's	Wittgenstein's
1.				FALSE
2.				
3.			FALSE	
4.				
5.	TRUE			
6.				
7.				
8.		TRUE		
9.				
10.				

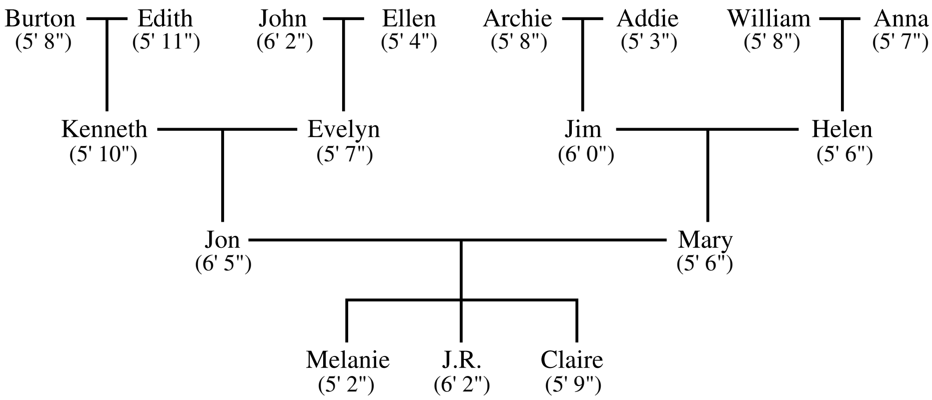

Figure 9.1: A family tree, with heights.

9.23 ✎ Consider the first-order language with function symbols mother and father, plus names for each of the people shown in the family tree in Figure 9.1. Here are some atomic wffs, each with a single free variable x. For each, pick a person for x that satisfies the wff, if you can. If there is no such person indicated in the family tree, say so.

1. mother(x) = ellen
2. father(x) = jon
3. mother(father(x)) = mary
4. father(mother(x)) = john
5. mother(father(x)) = addie
6. father(mother(father(x))) = john
7. father(father(mother(x))) = archie
8. father(father(jim)) = x
9. father(father(mother(claire))) = x
10. mother(mother(mary)) = mother(x)

9.24 ✎ Again using Figure 9.1, figure out which of the sentences listed below are true. Assume that the domain of discourse consists of the people listed in the family tree.

1. ∃x Taller(x, mother(x))
2. ∀x Taller(father(x), mother(x))
3. ∃y Taller(mother(mother(y)), mother(father(y)))
4. ∀z [z ≠ father(claire) → Taller(father(claire), z)]
5. ∀x [Taller(x, father(x)) → Taller(x, claire)]

9.25 Assume you are working in an extension of the first-order language of arithmetic with the additional predicates Even(x) and Prime(x). Express the following in this language, explicitly using the function symbol ×, as in $z \times z$, rather than z^2. Note that you do not have a predicate Square(x).

1. No square is prime.
2. Some square is odd.
3. The square of any prime is prime.
4. The square of any prime other than 2 is odd.
5. The square of any number greater than 1 is greater than the number itself.

Submit your sentence file.

Section 9.8
Alternative notation

The notation we have been using for the quantifiers is currently the most popular. An older notation that is still in some use employs (x) for ∀x. Thus, for example, in this notation our

$$\forall x\,[\mathsf{Tet}(x) \rightarrow \mathsf{Small}(x)]$$

would be written:

$$(x)\,[\mathsf{Tet}(x) \rightarrow \mathsf{Small}(x)]$$

Another notation that is occasionally used exploits the similarity between universal quantification and conjunction by writing $\bigwedge x$ instead of ∀x. In this notation our sentence would be rendered:

$$\bigwedge x\,[\mathsf{Tet}(x) \rightarrow \mathsf{Small}(x)]$$

Finally, you will sometimes encounter the universal quantifier written Πx, as in:

$$\Pi x\,[\mathsf{Tet}(x) \rightarrow \mathsf{Small}(x)]$$

Similar variants of ∃x are in use. One version writes (∃x) or (Ex). Other versions write $\bigvee x$ or Σx. Thus the following are notational variants of one another.

$$\exists x\,[\mathsf{Cube}(x) \land \mathsf{Large}(x)]$$
$$(\mathsf{E}x)[\mathsf{Cube}(x) \land \mathsf{Large}(x)]$$
$$\bigvee x\,[\mathsf{Cube}(x) \land \mathsf{Large}(x)]$$
$$\Sigma x\,[\mathsf{Cube}(x) \land \mathsf{Large}(x)]$$

Remember

The following table summarizes the alternative notations.

Our notation	Common equivalents
$\neg P$	$\sim P$, $\overline{P}$, $!P$, Np
$P \land Q$	$P\&Q$, $P\&\&Q$, $P \cdot Q$, PQ, Kpq
$P \lor Q$	$P \mid Q$, $P \parallel Q$, Apq
$P \to Q$	$P \supset Q$, Cpq
$P \leftrightarrow Q$	$P \equiv Q$, Epq
$\forall x\, S(x)$	$(x)S(x)$, $\bigwedge x\, S(x)$, $\Pi x\, S(x)$
$\exists x\, S(x)$	$(\exists x)S(x)$, $(Ex)S(x)$, $\bigvee x\, S(x)$, $\Sigma x\, S(x)$

Exercises

9.26 (Overcoming dialect differences) The following are all sentences of FOL. But they're in different dialects. Start a new sentence file in Tarski's World and translate them into our dialect.

1. $\sim (x)(P(x) \supset Q(x))$
2. $\Sigma y((P(y) \equiv \overline{Q(y)}) \,\&\, R(y))$
3. $\overline{\bigwedge x\, P(x)} \equiv \bigvee x\, \overline{P(x)}$

Exercises

10.1 ✎ For each of the following, use the truth-functional form algorithm to annotate the sentence and determine its form. Then classify the sentence as (a) a tautology, (b) a logical truth but not a tautology, or (c) not a logical truth. (If your answer is (a), feel free to use the **Taut Con** routine in Fitch to check your answer.)

1. $\forall x\, x = x$
2. $\exists x\, \mathsf{Cube}(x) \rightarrow \mathsf{Cube}(a)$
3. $\mathsf{Cube}(a) \rightarrow \exists x\, \mathsf{Cube}(x)$
4. $\forall x\, (\mathsf{Cube}(x) \land \mathsf{Small}(x)) \rightarrow \forall x\, (\mathsf{Small}(x) \land \mathsf{Cube}(x))$
5. $\forall v\, (\mathsf{Cube}(v) \leftrightarrow \mathsf{Small}(v)) \leftrightarrow \neg\neg\forall v\, (\mathsf{Cube}(v) \leftrightarrow \mathsf{Small}(v))$
6. $\forall x\, \mathsf{Cube}(x) \rightarrow \neg\exists x\, \neg\mathsf{Cube}(x)$
7. $[\forall z\, (\mathsf{Cube}(z) \rightarrow \mathsf{Large}(z)) \land \mathsf{Cube}(b)] \rightarrow \mathsf{Large}(b)$
8. $\exists x\, \mathsf{Cube}(x) \rightarrow (\exists x\, \mathsf{Cube}(x) \lor \exists y\, \mathsf{Dodec}(y))$
9. $(\exists x\, \mathsf{Cube}(x) \lor \exists y\, \mathsf{Dodec}(y)) \rightarrow \exists x\, \mathsf{Cube}(x)$
10. $[(\forall u\, \mathsf{Cube}(u) \rightarrow \forall u\, \mathsf{Small}(u)) \land \neg\forall u\, \mathsf{Small}(u)] \rightarrow \neg\forall u\, \mathsf{Cube}(u)$

Turn in your answers by filling in a table of the following form:

	Annotated sentence	Truth-functional form	a/b/c
1.			
⋮			

In the following six exercises, use the truth-functional form algorithm to annotate the argument. Then write out its truth-functional form. Finally, assess whether the argument is (a) tautologically valid, (b) logically but not tautologically valid, or (c) invalid. Feel free to check your answers with **Taut Con**. *(Exercises 10.6 and 10.7 are, by the way, particularly relevant to the proof of the Completeness Theorem for $\mathcal{F}$ given in Chapter 19.)*

10.2 ✎

$\mathsf{Cube}(a) \land \mathsf{Cube}(b)$
$\mathsf{Small}(a) \land \mathsf{Large}(b)$

$\exists x\, (\mathsf{Cube}(x) \land \mathsf{Small}(x)) \land \exists x\, (\mathsf{Cube}(x) \land \mathsf{Large}(x))$

10.3 ✎

$\forall x\, \mathsf{Cube}(x) \rightarrow \exists y\, \mathsf{Small}(y)$
$\neg\exists y\, \mathsf{Small}(y)$

$\exists x\, \neg\mathsf{Cube}(x)$

10.4 ✎

$\forall x\, \mathsf{Cube}(x) \rightarrow \exists y\, \mathsf{Small}(y)$
$\neg\exists y\, \mathsf{Small}(y)$

$\neg\forall x\, \mathsf{Cube}(x)$

10.5 ✎*

$\forall x\, (\mathsf{Tet}(x) \rightarrow \mathsf{LeftOf}(x, b)) \vee \forall x\, (\mathsf{Tet}(x) \rightarrow \mathsf{RightOf}(x, b))$
$\exists x\, (\mathsf{Tet}(x) \wedge \mathsf{SameCol}(x, b)) \rightarrow \neg\forall x\, (\mathsf{Tet}(x) \rightarrow \mathsf{LeftOf}(x, b))$
$\forall x\, (\mathsf{Tet}(x) \rightarrow \mathsf{RightOf}(x, b)) \rightarrow \neg\exists x\, (\mathsf{Tet}(x) \wedge \mathsf{SameCol}(x, b))$

$\neg\exists x\, (\mathsf{Tet}(x) \wedge \mathsf{SameCol}(x, b))$

10.6 ✎*

$\exists x\, (\mathsf{Cube}(x) \wedge \mathsf{Large}(x)) \rightarrow (\mathsf{Cube}(c) \wedge \mathsf{Large}(c))$
$\mathsf{Tet}(c) \rightarrow \neg\mathsf{Cube}(c)$
$\mathsf{Tet}(c)$

$\forall x\, \neg(\mathsf{Cube}(x) \wedge \mathsf{Large}(x))$

10.7 ✎*

$\exists x\, (\mathsf{Cube}(x) \wedge \mathsf{Large}(x)) \rightarrow (\mathsf{Cube}(c) \wedge \mathsf{Large}(c))$
$\forall x\, \neg(\mathsf{Cube}(x) \wedge \mathsf{Large}(x)) \leftrightarrow \neg\exists x\, (\mathsf{Cube}(x) \wedge \mathsf{Large}(x))$
$\mathsf{Tet}(c) \rightarrow \neg\mathsf{Cube}(c)$
$\mathsf{Tet}(c)$

$\forall x\, \neg(\mathsf{Cube}(x) \wedge \mathsf{Large}(x))$

[In 10.6 and 10.7, we could think of the first premise as a way of introducing a new constant, c, by means of the assertion: *Let the constant* c *name a large cube, if there are any; otherwise, it may name any object.* Sentences of this sort are called *Henkin witnessing axioms,* and are put to important use in proving completeness for $\mathcal{F}$. The arguments show that if a constant introduced in this way ends up naming a tetrahedron, it can only be because there aren't any large cubes.]

Section 10.2

First-order validity and consequence

When we first discussed the intuitive notions of logical truth and logical consequence, we appealed to the idea of a logically possible circumstance. We described a logically valid argument, for example, as one whose conclusion is true in every possible circumstance in which all the premises are true. When we needed more precision than this description allowed, we introduced truth tables and the concepts of tautology and tautological consequence. These

Exercises

10.8 If you skipped the **You try it** section, go back and do it now. Submit the file Proof FO Con 1.

10.9 Open Carnap's Sentences and Bolzano's World.

1. Paraphrase each sentence in clear, colloquial English and verify that it is true in the given world.
2. For each sentence, decide whether you think it is a logical truth. If it isn't, build a world in which the sentence comes out false and save it as World 10.9.x, where x is the number of the sentence. [Hint: You should be able to falsify three of these sentences.]
3. Which of these sentences are first-order validities? [Hint: Three are.]
4. For the remaining four sentences (those that are logical truths but not first-order validities), apply the Replacement Method to come up with first-order counterexamples. Make sure you describe both your interpretations of the predicates and the falsifying circumstance.

Turn in your answers to parts 1, 3, and 4; submit the worlds you build in part 2.

Each of the following arguments is valid. Some of the conclusions are (a) tautological consequences of the premises, some are (b) first-order consequences that are not tautological consequences, and some are (c) logical consequences that are not first-order consequences. Use the truth-functional form algorithm and the replacement method to classify each argument. You should justify your classifications by turning in (a) the truth-functional form of the argument, (b) the truth-functional form and the argument with nonsense predicates substituted, or (c) the truth-functional form, the nonsense argument, and a first-order counterexample.

10.10

Cube(a) ∧ Cube(b)
Small(a) ∧ Large(b)
——
∃x (Cube(x) ∧ Small(x)) ∧ ∃x (Cube(x) ∧ Large(x))

10.11

Cube(a) ∧ Cube(b)
Small(a) ∧ Large(b)
——
∃x (Cube(x) ∧ Large(x) ∧ ¬Smaller(x, x))

10.12

∀x Cube(x) → ∃y Small(y)
¬∃y Small(y)
——
∃x ¬Cube(x)

10.13

∀x Cube(x) → ∃y Small(y)
¬∃y Small(y)
——
¬∀x Cube(x)

10.14

$$\begin{array}{|l} \mathsf{Cube(a)} \\ \mathsf{Dodec(b)} \\ \hline \neg(\mathsf{a} = \mathsf{b}) \end{array}$$

10.15

$$\begin{array}{|l} \mathsf{Cube(a)} \\ \neg\mathsf{Cube(b)} \\ \hline \neg(\mathsf{a} = \mathsf{b}) \end{array}$$

10.16

$$\begin{array}{|l} \mathsf{Cube(a)} \\ \neg\mathsf{Cube(a)} \\ \hline \neg(\mathsf{a} = \mathsf{b}) \end{array}$$

10.17

$$\begin{array}{|l} \forall \mathsf{x}\,(\mathsf{Dodec(x)} \to \neg\mathsf{SameCol(x, c)}) \\ \hline \neg\mathsf{Dodec(c)} \end{array}$$

10.18

$$\begin{array}{|l} \forall \mathsf{z}\,(\mathsf{Small(z)} \leftrightarrow \mathsf{Cube(z)}) \\ \mathsf{Cube(d)} \\ \hline \mathsf{Small(d)} \end{array}$$

10.19

$$\begin{array}{|l} \forall \mathsf{z}\,(\mathsf{Small(z)} \to \mathsf{Cube(z)}) \\ \forall \mathsf{w}\,(\mathsf{Cube(w)} \to \mathsf{LeftOf(w, c)}) \\ \hline \neg\exists \mathsf{y}\,(\mathsf{Small(y)} \land \neg\mathsf{LeftOf(y, c)}) \end{array}$$

Section 10.3

First-order equivalence and DeMorgan's laws

There are two ways in which we can apply what we learned about tautological equivalence to first-order sentences. First of all, if you apply the truth-functional form algorithm to a pair of sentences and the resulting forms are tautologically equivalent, then of course the original sentences are first-order equivalent. For example, the sentence:

$$\neg(\exists \mathsf{x}\, \mathsf{Cube(x)} \land \forall \mathsf{y}\, \mathsf{Dodec(y)})$$

is tautologically equivalent to:

$$\neg\exists \mathsf{x}\, \mathsf{Cube(x)} \lor \neg\forall \mathsf{y}\, \mathsf{Dodec(y)}$$

When you apply the truth-functional form algorithm, you see that this is just an instance of one of DeMorgan's laws.

But it turns out that we can also apply DeMorgan, and similar principles, inside the scope of quantifiers. Let's look at an example involving the Law of Contraposition. Consider the sentences:

$$\forall \mathsf{x}\,(\mathsf{Cube(x)} \to \mathsf{Small(x)})$$
$$\forall \mathsf{x}\,(\neg\mathsf{Small(x)} \to \neg\mathsf{Cube(x)})$$

$$\begin{array}{rcl} \neg\forall x\,(P(x) \rightarrow Q(x)) & \Leftrightarrow & \neg\forall x\,(\neg P(x) \lor Q(x)) \\ & \Leftrightarrow & \exists x\,\neg(\neg P(x) \lor Q(x)) \\ & \Leftrightarrow & \exists x\,(\neg\neg P(x) \land \neg Q(x)) \\ & \Leftrightarrow & \exists x\,(P(x) \land \neg Q(x)) \end{array}$$

The first step uses the equivalence of $P(x) \rightarrow Q(x)$ and $\neg P(x) \lor Q(x)$. The second and third steps use DeMorgan's laws, first one of the quantifier versions, and then one of the Boolean versions. The last step uses the double negation law applied to $\neg\neg P(x)$.

A similar chain of equivalences shows that the negation of *Some P's are Q's* is equivalent to *No P's are Q's*:

$$\neg\exists x\,(P(x) \land Q(x)) \quad \Leftrightarrow \quad \forall x\,(P(x) \rightarrow \neg Q(x))$$

We leave the demonstration of this as an exercise.

Remember

(DeMorgan laws for quantifiers) For any wff $P(x)$:

1. $\neg\forall x\,P(x) \;\Leftrightarrow\; \exists x\,\neg P(x)$
2. $\neg\exists x\,P(x) \;\Leftrightarrow\; \forall x\,\neg P(x)$

Exercises

10.20 ✎ Give a chain of equivalences showing that the negation of *Some P's are Q's* ($\neg\exists x\,(P(x) \land Q(x))$) is equivalent to *No P's are Q's* ($\forall x\,(P(x) \rightarrow \neg Q(x))$).

10.21 ✎ Open DeMorgan's Sentences 2. This file contains six sentences, but each of sentences 4, 5, and 6 is logically equivalent to one of the first three. Without looking at what the sentences say, see if you can figure out which is equivalent to which by opening various world files and evaluating the sentences. (You should be able to figure this out from Ackermann's, Bolzano's, and Claire's Worlds, plus what we've told you.) Once you think you've figured out which are equivalent to which, write out three equivalence chains to prove you're right. Turn these in to your instructor.

10.22 ($\forall$ versus $\wedge$) We pointed out the similarity between $\forall$ and $\wedge$, as well as that between $\exists$ and $\vee$. But we were careful not to claim that the universally quantified sentence was logically equivalent to the analogous conjunction. This problem will show you why we did not make this claim.

- Open Church's Sentences and Ramsey's World. Evaluate the sentences in this world. You will notice that the first two sentences have the same truth value, as do the second two.
- Modify Ramsey's World in any way you like, but do not add or delete objects, and do not change the names used. Verify that the first two sentences always have the same truth values, as do the last two.
- Now add one object to the world. Adjust the objects so that the first sentence is false, the second and third true, and the last false. Submit your work as World 10.22. This world shows that the first two sentences are not logically equivalent. Neither are the last two.

Section 10.4

Other quantifier equivalences

The quantifier DeMorgan laws tell us how quantifiers interact with negation. Equally important is the question of how quantifiers interact with conjunction and disjunction. The laws governing this interaction, though less interesting than DeMorgan's, are harder to remember, so you need to pay attention!

quantifiers and Boolean connectives

First of all, notice that $\forall x\, (P(x) \wedge Q(x))$, which says that everything is both P and Q, is logically equivalent to $\forall x\, P(x) \wedge \forall x\, Q(x)$, which says that everything is P and everything is Q. These are just two different ways of saying that every object in the domain of discourse has both properties P and Q. By contrast, $\forall x\, (P(x) \vee Q(x))$ is *not* logically equivalent to $\forall x\, P(x) \vee \forall x\, Q(x)$. For example, the sentence $\forall x\, (\mathsf{Cube}(x) \vee \mathsf{Tet}(x))$ says that everything is either a cube or a tetrahedron, but the sentence $\forall x\, \mathsf{Cube}(x) \vee \forall x\, \mathsf{Tet}(x)$ says that either everything is a cube or everything is a tetrahedron, clearly a very different kettle of fish. We summarize these two observations, positive and negative, as follows:

$$\forall x\, (P(x) \wedge Q(x)) \quad \Leftrightarrow \quad \forall x\, P(x) \wedge \forall x\, Q(x)$$
$$\forall x\, (P(x) \vee Q(x)) \quad \not\Leftrightarrow \quad \forall x\, P(x) \vee \forall x\, Q(x)$$

Similar observations hold with $\exists$, $\vee$, and $\wedge$, except that it works the other way around. The claim that there is some object that is either P or Q, $\exists x\, (P(x) \vee Q(x))$, is logically equivalent to the claim that something is

Remember

1. (Pushing quantifiers past $\land$ and $\lor$) For any wffs $\mathsf{P(x)}$ and $\mathsf{Q(x)}$:
   (a) $\forall x\,(P(x) \land Q(x)) \;\Leftrightarrow\; \forall x\,P(x) \land \forall x\,Q(x)$
   (b) $\exists x\,(P(x) \lor Q(x)) \;\Leftrightarrow\; \exists x\,P(x) \lor \exists x\,Q(x)$
2. (Null quantification) For any wff P in which x is not free:
   (a) $\forall x\,P \;\Leftrightarrow\; P$
   (b) $\exists x\,P \;\Leftrightarrow\; P$
   (c) $\forall x\,(P \lor Q(x)) \;\Leftrightarrow\; P \lor \forall x\,Q(x)$
   (d) $\exists x\,(P \land Q(x)) \;\Leftrightarrow\; P \land \exists x\,Q(x)$
3. (Replacing bound variables) For any wff $\mathsf{P(x)}$ and variable y that does not occur in $\mathsf{P(x)}$:
   (a) $\forall x\,P(x) \;\Leftrightarrow\; \forall y\,P(y)$
   (b) $\exists x\,P(x) \;\Leftrightarrow\; \exists y\,P(y)$

Exercises

10.23 (Null quantification) Open Null Quantification Sentences. In this file you will find sentences in the odd numbered slots. Notice that each sentence is obtained by putting a quantifier in front of a sentence in which the quantified variable is not free.

1. Open Godel's World and evaluate the truth of the first sentence. Do you understand why it is false? Repeatedly play the game committed to the truth of this sentence, each time choosing a different block when your turn comes around. Not only do you always lose, but your choice has no impact on the remainder of the game. Frustrating, eh?
2. Check the truth of the remaining sentences and make sure you understand why they have the truth values they do. Play the game a few times on the second sentence, committed to both true and false. Notice that neither your choice of a block (when committed to false) nor Tarski's World's choice (when committed to true) has any effect on the game.

3. In the even numbered slots, write the sentence from which the one above it was obtained. Check that the even and odd numbered sentences have the same truth value, no matter how you modify the world. This is because they are logically equivalent. Save and submit your sentence file.

Some of the following biconditionals are logical truths (which is the same as saying that the two sides of the biconditional are logically equivalent); some are not. If you think the biconditional is a logical truth, create a file with Fitch, enter the sentence, and check it using **FO Con**. *If the sentence is not a logical truth, create a world in Tarski's World in which it is false. Submit the file you create.*

10.24 $(\forall x\, \mathsf{Cube}(x) \lor \forall x\, \mathsf{Dodec}(x)) \leftrightarrow \forall x\, (\mathsf{Cube}(x) \lor \mathsf{Dodec}(x))$

10.25 $\neg\exists z\, \mathsf{Small}(z) \leftrightarrow \exists z\, \neg\mathsf{Small}(z)$

10.26 $\forall x\, \mathsf{Tet}(b) \leftrightarrow \exists w\, \mathsf{Tet}(b)$

10.27 $\exists w\, (\mathsf{Dodec}(w) \land \mathsf{Large}(w)) \leftrightarrow (\exists w\, \mathsf{Dodec}(w) \land \exists w\, \mathsf{Large}(w))$

10.28 $\exists w\, (\mathsf{Dodec}(w) \land \mathsf{Large}(b)) \leftrightarrow (\exists w\, \mathsf{Dodec}(w) \land \mathsf{Large}(b))$

10.29 $\neg\forall x\, (\mathsf{Cube}(x) \to (\mathsf{Small}(x) \lor \mathsf{Large}(x))) \leftrightarrow \exists z\, (\mathsf{Cube}(z) \land \neg\mathsf{Small}(z) \land \neg\mathsf{Large}(z))$

Section 10.5

The axiomatic method

As we will see in the coming chapters, first-order consequence comes much closer to capturing the logical consequence relation of ordinary language than does tautological consequence. This will be apparent from the kinds of sentences that we can translate into the quantified language and from the kinds of inference that turn out to be first-order valid.

Still, we have already encountered several arguments that are intuitively valid but not first-order valid. Let's look at an example where the replacement method reveals that the conclusion is not a first-order consequence of the premises:

$$\begin{array}{|l} \forall x\, (\mathsf{Cube}(x) \leftrightarrow \mathsf{SameShape}(x, c)) \\ \hline \mathsf{Cube}(c) \end{array}$$

Using the replacement method, we substitute meaningless predicate symbols, say P and Q, for the predicates Cube and SameShape. The result is

$$\begin{array}{|l} \forall x\, (P(x) \leftrightarrow Q(x, c)) \\ \hline P(c) \end{array}$$

that follows from the axioms by valid methods of inference is on just as firm a footing as the axioms from which we start.

Exercises

10.30 If you skipped the **You try it** section, go back and do it now. Submit the file Proof Axioms 1.

10.31 Suppose we state our four basic shape axioms in the following schematic form:

1. ¬∃x (R(x) ∧ P(x))
2. ¬∃x (P(x) ∧ Q(x))
3. ¬∃x (Q(x) ∧ R(x))
4. ∀x (P(x) ∨ Q(x) ∨ R(x))

We noted that any valid argument involving just the three shape predicates remains valid when you substitute other predicates, like the Tarski's World size predicates, that satisfy these axioms. Which of the following triplets of properties satisfy the axioms in the indicated domain (that is, make them true when you substitute them for P, Q, and R)? If they don't, say which axioms fail and why.

1. *Red, yellow,* and *blue* in the domain of automobiles.
2. *Entirely red, entirely yellow,* and *entirely blue* in the domain of automobiles.
3. *Small, medium,* and *large* in the domain of Tarski's World blocks.
4. *Small, medium,* and *large* in the domain of physical objects.

[Note: Your answers in some cases will depend on how you are construing the predicates. The important thing is that you explain your interpretations clearly, and how the interpretations lead to the success or failure of the axioms.]

Section 10.6

Lemmas

One important feature of the axiomatic method is that as theorems are proved, they become available as new facts that can be used in later proofs. If S has been shown to be a consequence of the axioms, then S can be used in any proof which employs those same axioms. After all, we could just re-prove S in the new proof, since all of its premises are available in the new proof, and then cite S in subsequent inference steps.

For example, suppose that you have proved that

¬∃x(Cube(x) ∧ Dodec(x) ∧ Tet(x))

In general, if a substitution like this can be found that associates a general formula in the lemma with a formula in the citations and goal, then the step will check out.

. . . . ***Congratulations***

Just as for predicate symbols and formulae, you can use the general constant symbols $n_1, \ldots, n_2$ in your lemmas to say that the lemma does not concern specific constant symbols. A lemma such as $\mathsf{Tet}(a) \rightarrow \neg\mathsf{Cube}(a)$ can be used only to match that exact formula, but $\mathsf{Tet}(n_1) \rightarrow \neg\mathsf{Cube}(n_1)$ could be used to prove $\mathsf{Tet}(a) \rightarrow \neg\mathsf{Cube}(a)$, $\mathsf{Tet}(f) \rightarrow \neg\mathsf{Cube}(f)$, or even $\mathsf{Tet}(n_2) \rightarrow \neg\mathsf{Cube}(n_2)$.

Remember

1. Lemmas are just proofs that you have completed that are being used to justify a step in another proof.
2. Fitch's lemma rule requires that each cited formula matches a premise in the lemma file, and the derived formula matches the goal of the lemma file.
3. The letters P, Q etc; in a lemma file will match any formula.
4. The constant symbols $n_1, \ldots, n_9$ in a lemma file will match any constant symbol.

Exercises

10.32 ➶ If you skipped the **You try it** sections, go back and do them now. Submit the files Proof Lemma Example 1, Proof Lemma Example 2 and Proof Lemma Example 3.

10.33 ✎ Which of the following arguments can be justified by a single application of Lemma 3? For each one that can be justified, turn in the substitution of formulas that is required to justify the application. For each argument that cannot be justified, explain why not.

1. | Cube(a) ∨ Cube(b)
   | Cube(b) → SameSize(a, b)
   | Cube(a) → SameShape(a, b)
   |—
   | SameSize(a, b) ∨ SameShape(a, b)

2. | Tet(a) ∨ Tet(b)
   | Tet(b) → Larger(e, f)
   | Tet(a) → Smaller(f, e)
   |—
   | Smaller(f, e) ∨ Larger(e, f)

3. | (Dodec(d) ∧ Small(d)) → Larger(e, f)
   | Large(e) → (Adjoins(a, b) ∨ Adjoins(a, c))
   | Large(e) ∨ (Dodec(d) ∧ Small(d))
   |—
   | (Adjoins(a, b) ∨ Adjoins(a, c)) ∨ Larger(e, f)

4. | Tet(e)
   | Tet(e) → Small(e)
   | Tet(e) → SameSize(e, e)
   |—
   | Small(e) ∨ SameSize(e, e)

5. | LeftOf(a, b) ∨ Smaller(a, b)
   | Smaller(a, b) → P
   | LeftOf(a, b) → Q
   |—
   | Q ∨ P

10.34 Prove a single lemma which can be used to complete each of the proofs in the files Exercise 10.34.1 and Exercise 10.34.2 in one step using the **Lemma** rule. Submit the lemma file as Proof 10.34.

► 5. The second sentence in the file looks for all the world like it says there are two cubes. But it doesn't. Delete all but one cube in the world and check to see that it's still true. Play the game committed to FALSE and see what happens.

► 6. See if you can modify the second sentence so it is false in a world with only one cube, but true if there are two or more. (Use $\neq$ like we did above.) Save the modified sentences as Sentences Multiple 1.

. . . . ***Congratulations***

identity and variables

In general, to say that every *pair* of distinct objects stands in some relation, you need a sentence of the form $\forall x \forall y (x \neq y \rightarrow \ldots)$, and to say that there are *two* objects with a certain property, you need a sentence of the form $\exists x \exists y (x \neq y \land \ldots)$. Of course, other parts of the sentence often guarantee the distinctness for you. For example if you say that every tetrahedron is larger than every cube:

$$\forall x \forall y ((\mathsf{Tet}(x) \land \mathsf{Cube}(y)) \rightarrow \mathsf{Larger}(x, y))$$

then the fact that x must be a tetrahedron and y a cube ensures that your claim says what you intended.

Remember

When evaluating a sentence with multiple quantifiers, don't fall into the trap of thinking that distinct variables range over distinct objects. In fact, the sentence $\forall x \forall y\, P(x, y)$ logically implies $\forall x\, P(x, x)$, and the sentence $\exists x\, P(x, x)$ logically implies $\exists x \exists y\, P(x, y)$!

Exercises

11.1 If you skipped the **You try it** section, go back and do it now. Submit the file Sentences Multiple 1.

11.2 (Simple multiple quantifier sentences) The file Frege's Sentences contains 14 sentences; the first seven begin with a pair of existential quantifiers, the second seven with a pair of universal quantifiers. Go through the sentences one by one, evaluating them in Peirce's World. Though you probably won't have any trouble understanding these sentences, don't forget to use the game if you do. When you understand all the sentences, modify the size and location of a single block so that the first seven sentences are true and the second seven false. Submit the resulting world.

11.3 (Getting fancier) Open up Peano's World and Peano's Sentences. The sentence file contains 30 assertions that Alex made about this world. Evaluate Alex's claims. If you have trouble with any, play the game (several times if necessary) until you see where you are going wrong. Then change each of Alex's false claims into a true claim. If you can make the sentence true by adding a clause of the form $x \neq y$, do so. Otherwise, see if you can turn the false claim into an interesting truth: don't just add a negation sign to the front of the sentence. Submit your corrected list of sentences.

11.4 (Describing a world) Let's try our hand describing a world using multiple quantifiers. Open Finsler's World and start a new sentence file.

1. Notice that all the small blocks are in front of all the large blocks. Use your first sentence to say this.
2. With your second sentence, point out that there's a cube that is larger than a tetrahedron.
3. Next, say that all the cubes are in the same column.
4. Notice, however, that this is not true of the tetrahedra. So write the same sentence about the tetrahedra, but put a negation sign out front.
5. Every cube is also in a different row from every other cube. Say this.
6. Again, this isn't true of the tetrahedra, so say that it's not.
7. Notice there are different tetrahedra that are the same size. Express this fact.
8. But there aren't different cubes of the same size, so say that, too.

Are all your translations true in Finsler's World? If not, try to figure out why. In fact, play around with the world and see if your first-order sentences always have the same truth values as the claims you meant to express. Check them out in Konig's World, where all of the original claims are false. Are your sentences all false? When you think you've got them right, submit your sentence file.

11.5 (Building a world) Open Ramsey's Sentences. Build a world in which sentences 1–10 are all true at once (ignore sentences 11–20 for now). These first ten sentences all make either *particular* claims (that is, they contain no quantifiers) or *existential* claims (that is, they assert that things of a certain sort exist). Consequently, you could make them true by successively adding objects to the world. But part of the exercise is to make them all true *with as few objects as possible.* You should be able to do it with a total of six objects. So rather than adding objects for each new sentence, only add new objects when absolutely necessary. Again, be sure to go back and check that all the sentences are true when you are finished. Submit your world as World 11.5. [Hint: To make all the sentences true with six blocks, you will have to watch out for some intentionally misleading implicatures. For example, one of the objects will have to have two names.]

11.6 (Modifying the world) Sentences 11-20 of Ramsey's Sentences all make *universal* claims. That is, they all say that every object in the world has some property or other. Check to see whether the world you have built in Exercise 11.5 satisfies the universal claims expressed by these sentences. If not, modify the world so it makes all 20 sentences true at once. Submit your modified world as World 11.6. (Make sure you submit both World 11.5 and World 11.6 to get credit for both exercises.)

11.7 (Block parties) The interaction of quantifiers and negation gives rise to subtleties that can be pretty confusing. Open Löwenheim's Sentences, which contains eight sentences divided into two sets. Suppose we imagine a column containing blocks to be a *party* and think of the blocks in the column as the attendees. We'll say a party is *lonely* if there's only one block attending it, and say a party is *exclusive* if there's any block who's not there (i.e., who's in another column).

1. Using this terminology, give simple and clear English renditions of each of the sentences. For example, sentence 2 says *some of the parties are not lonely*, and sentence 7 says *there's only one party*. You'll find sentences 4 and 9 the hardest to understand. Construct a lot of worlds to see what they mean.

2. With the exception of 4 and 9, all of the sentences are first-order equivalent to other sentences on the list, or to negations of other sentences (or both). Which sentences are 3 and 5 equivalent to? Which sentences do 3 and 5 negate?

3. Sentences 4 and 9 are logically independent: it's possible for the two to have any pattern of truth values. Construct four worlds: one in which both are true (World 11.7.1), one in which 4 is true and 9 false (World 11.7.2), one in which 4 is false and 9 true (World 11.7.3), and one in which both are false (World 11.7.4).

Submit the worlds you've constructed and turn the remaining answers in to your instructor.

Section 11.2

Mixed quantifiers

Ready to start juggling with both hands? We now turn to the important case in which universal and existential quantifiers get mixed together. Let's start with the following sentence:

$$\forall x\, [\mathsf{Cube}(x) \rightarrow \exists y\, (\mathsf{Tet}(y) \land \mathsf{LeftOf}(x, y))]$$

This sentence shouldn't throw you. It has the overall Aristotelian form $\forall x\, [\mathsf{P}(x) \rightarrow \mathsf{Q}(x)]$, which we have seen many times before. It says that every cube has some property or other. What property? The property expressed

With mixed quantifiers and identity, we can say quite a bit more. For example, consider the sentence

$$\exists x\,(\mathsf{Cube}(x) \land \forall y\,(\mathsf{Cube}(y) \rightarrow y = x))$$

exactly one

This says that there is a cube, and furthermore every cube is identical to it. Some cube, in other words, is the *only* cube. Thus, this sentence will be true if and only if there is exactly one cube. There are many ways of saying things like this in FOL; we'll run across others in the exercises. We discuss numerical claims more systematically in Chapter 14.

Remember

When you are dealing with mixed quantifiers, the order is very important. $\forall x\,\exists y\,R(x, y)$ is not logically equivalent to $\exists y\,\forall x\,R(x, y)$.

Exercises

11.8 If you skipped the **You try it** section, go back and do it now. Submit the file World Mixed 1.

11.9 (Simple mixed quantifier sentences) Open Hilbert's Sentences and Peano's World. Evaluate the sentences one by one, playing the game if an evaluation surprises you. Once you understand the sentences, modify the false ones by adding a single negation sign so that they come out true. The catch is that you aren't allowed to add the negation sign to the front of the sentence! Add it to an atomic formula, if possible, and try to make the claim nonvacuously true. (This won't always be possible.) Make sure you understand both why the original sentence is false and why your modified sentence is true. When you're done, submit your sentence list with the changes.

11.10 (Mixed quantifier sentences with identity) Open Leibniz's World and use it to evaluate the sentences in Leibniz's Sentences. Make sure you understand all the sentences and follow any instructions in the file. Submit your modified sentence list.

11.11 (Building a world) Create a world in which all ten sentences in Arnault's Sentences are true. Submit your world.

11.12 (Name that object) Open Carroll's World and Hercule's Sentences. Try to figure out which objects have names, and what they are. You should be able to figure this out from the sentences, all of which are true. Once you have come to your conclusion, add the names to the objects and check to see if all the sentences are true. Submit your modified world.

The remaining three exercises all have to do with the sentences in the file Buridan's Sentences *and build on one another.*

11.13 (Building a world) Open Buridan's Sentences. Build a world in which all ten sentences are true. Submit your world.

11.14 (Consequence) These two English sentences are consequences of the ten sentences in Buridan's Sentences.

1. *There are no cubes.*
2. *There is exactly one large tetrahedron.*

Because of this, they must be true in any world in which Buridan's sentences are all true. So of course they must be true in World 11.13, no matter how you built it.

- Translate the two sentences, adding them to the list in Buridan's Sentences. Name the expanded list Sentences 11.14. Verify that they are all true in World 11.13.
- Modify the world by adding a cube. Try placing it at various locations and giving it various sizes to see what happens to the truth values of the sentences in your file. One or more of the original ten sentences will always be false, though different ones at different times. Find a world in which only one of the original ten sentences is false and name it World 11.14.1.
- Next, get rid of the cube and add a second large tetrahedron. Again, move it around and see what happens to the truth values of the sentences. Find a world in which only one of the original ten sentences is false and name it World 11.14.2.

Submit your sentence file and two world files.

11.15 ★ (Independence) Show that the following sentence is independent of those in Buridan's Sentences, that is, neither it nor its negation is a consequence of those sentences.

$$\exists x\, \exists y\, (x \neq y \land \mathsf{Tet}(x) \land \mathsf{Tet}(y) \land \mathsf{Medium}(x) \land \mathsf{Medium}(y))$$

You will do this by building two worlds, one in which this sentence is false (call this World 11.15.1) and one in which it is true (World 11.15.2)—but both of which make all of Buridan's sentences true.

Section 11.3
The step-by-step method of translation

When an English sentence contains more than one quantified noun phrase, translating it can become quite confusing unless you approach it in a very systematic way. It often helps to go through a few intermediate steps, treating the quantified noun phrases one at a time.

Suppose, for example, we wanted to translate the sentence *Each cube is to the left of a tetrahedron*. Here, there are two quantified noun phrases: *each cube* and *a tetrahedron*. We can start by dealing with the first noun phrase, temporarily treating the complex phrase *is-to-the-left-of-a-tetrahedron* as a single unit. In other words, we can think of the sentence as a single quantifier sentence, on the order of *Each cube is small*. The translation would look like this:

$$\forall x\,(\mathsf{Cube}(x) \rightarrow x \textit{ is-to-the-left-of-a-tetrahedron})$$

Of course, this is not a sentence in our language, so we need to translate the expression x *is-to-the-left-of-a-tetrahedron*. But we can think of this expression as a single quantifier sentence, at least if we pretend that x is a name. It has the same general form as the sentence ***b*** *is to the left of a tetrahedron*, and would be translated as

$$\exists y\,(\mathsf{Tet}(y) \land \mathsf{LeftOf}(x, y))$$

Substituting this in the above, we get the desired translation of the original English sentence:

$$\forall x\,(\mathsf{Cube}(x) \rightarrow \exists y\,(\mathsf{Tet}(y) \land \mathsf{LeftOf}(x, y)))$$

This is exactly the sentence with which we began our discussion of mixed quantifiers.

This step-by-step process really comes into its own when there are lots of quantifiers in a sentence. It would be very difficult for a beginner to translate a sentence like *No cube to the right of a tetrahedron is to the left of a larger dodecahedron* in a single blow. Using the step-by-step method makes it straightforward. Eventually, though, you will be able to translate quite complex sentences, going through the intermediate steps in your head.

Exercises

11.16 (Using the step-by-step method of translation)

- Open Montague's Sentences. This file contains expressions that are halfway between English and first-order logic. Our goal is to edit this file until it contains translations of the following English sentences. You should read the English sentence below, make sure you understand how we got to the halfway point, and then complete the translation by replacing the hyphenated expression with a wff of first-order logic.

    1. *Every cube is to the left of every tetrahedron.* [In the Sentence window, you see the halfway completed translation, together with some blanks that need to be replaced by wffs. Commented out below this, you will find an intermediate "sentence." Make sure you understand how we got to this intermediate stage of the translation. Then complete the translation by replacing the blank with

    $$\forall y\,(\mathsf{Tet}(y) \rightarrow \mathsf{LeftOf}(x, y))$$

       Once this is done, check to see if you have a well-formed sentence. Does it look like a proper translation of the original English? It should.]
    2. *Every small cube is in back of a large cube.*
    3. *Some cube is in front of every tetrahedron.*
    4. *A large cube is in front of a small cube.*
    5. *Nothing is larger than everything.*
    6. *Every cube in front of every tetrahedron is large.*
    7. *Everything to the right of a large cube is small.*
    8. *Nothing in back of a cube and in front of a cube is large.*
    9. *Anything with nothing in back of it is a cube.*
    10. *Every dodecahedron is smaller than some tetrahedron.*

  Save your sentences as Sentences 11.16.

- Open Peirce's World. Notice that all the English sentences are true in this world. Check to see that all of your translations are true as well. If they are not, see if you can figure out where you went wrong.

- Open Leibniz's World. Note that the English sentences 5, 6, 8, and 10 are true in this world, while the rest are false. Verify that your translations have the same truth values. If they don't, fix them.

- Open Ron's World. Here, the true sentences are 2, 3, 4, 5, and 8. Check that your translations have the right values, and correct them if they don't.

11.17 (More multiple quantifier sentences) Now, we will try translating some multiple quantifier sentences completely from scratch. You should try to use the step-by-step procedure.

- Start a new sentence file and translate the following English sentences.

  1. *Every tetrahedron is in front of every dodecahedron.*
  2. *No dodecahedron has anything in back of it.*
  3. *No tetrahedron is the same size as any cube.*
  4. *Every dodecahedron is the same size as some cube.*
  5. *Anything between two dodecahedra is a cube.* [Note: This use of *two* really can be paraphrased using *between a dodecahedron and a dodecahedron.*]
  6. *Every cube falls between two objects.*
  7. *Every cube with something in back of it is small.*
  8. *Every dodecahedron with nothing to its right is small.*
  9. (⋆) *Every dodecahedron with nothing to its right has something to its left.*
  10. *Any dodecahedron to the left of a cube is large.*

- Open Bolzano's World. All of the above English sentences are true in this world. Verify that all your translations are true as well.

- Now open Ron's World. The English sentences 4, 5, 8, 9, and 10 are true, but the rest are false. Verify that the same holds of your translations.

- Open Claire's World. Here you will find that the English sentences 1, 3, 5, 7, 9, and 10 are true, the rest false. Again, check to see that your translations have the appropriate truth value.

- Finally, open Peano's World. Notice that only sentences 8 and 9 are true. Check to see that your translations have the same truth values.

Section 11.4

Paraphrasing English

Some English sentences do not easily lend themselves to direct translation using the step-by-step procedure. With such sentences, however, it is often quite easy to come up with an English paraphrase that is amenable to the procedure. Consider, for example, *If a freshman takes a logic class, then he or she must be smart.* The step-by-step procedure does not work here. If we try to apply the procedure we would get something like

Remember

In translating from English to FOL, the goal is to get a sentence that has the same meaning as the original. This sometimes requires changes in the surface form of the sentence.

Exercises

11.18 (Sentences that need paraphrasing before translation) Translate the following sentences by first giving a suitable English paraphrase. Some of them are donkey sentences, so be careful.

1. *Only large objects have nothing in front of them.*
2. *If a cube has something in front of it, then it's small.*
3. *Every cube in back of a dodecahedron is also smaller than it.*
4. *If* **e** *is between two objects, then they are both small.*
5. *If a tetrahedron is between two objects, then they are both small.*

Open Ron's World. Recall that there are lots of hidden things in this world. Each of the above English sentences is true in this world, so the same should hold of your translations. Check to see that it does. Now open Bolzano's World. In this world, only sentence 3 is true. Check that the same holds of your translations. Next open Wittgenstein's World. In this world, only the English sentence 5 is true. Verify that your translations have the same truth values. Submit your sentence file.

11.19 (More sentences that need paraphrasing before translation) Translate the following sentences by first giving a suitable English paraphrase.

1. *Every dodecahedron is as large as every cube.* [Hint: Since we do not have anything corresponding to *as large as* (by which we mean at least as large as) in our language, you will first need to paraphrase this predicate using *larger than or same size as.*]
2. *If a cube is to the right of a dodecahedron but not in back of it, then it is as large as the dodecahedron.*
3. *No cube with nothing to its left is between two cubes.*
4. *The only large cubes are* ***b*** *and* ***c****.*
5. *At most* ***b*** *and* ***c*** *are large cubes.* [Note: There is a significant difference between this sentence and the previous one. This one does not imply that b and c are large cubes, while the previous sentence does.]

Open Ron's World. Each of the above English sentences is true in this world, so the same should hold of your translations. Check to see that it does. Now open Bolzano's World. In this world, only sentences 3 and 5 are true. Check that the same holds of your translations. Next open Wittgenstein's World. In this world, only the English sentences 2 and 3 are true. Verify that your translations have the same truth values. Submit your sentence file.

11.20 (More translations) The following English sentences are true in Godel's World. Translate them, and make sure your translations are also true. Then modify the world in various ways, and check that your translations track the truth value of the English sentence.

1. *Nothing to the left of **a** is larger than everything to the left of **b**.*
2. *Nothing to the left of **a** is smaller than anything to the left of **b**.*
3. *The same things are left of **a** as are left of **b**.*
4. *Anything to the left of **a** is smaller than something that is in back of every cube to the right of **b**.*
5. *Every cube is smaller than some dodecahedron but no cube is smaller than every dodecahedron.*
6. *If **a** is larger than some cube then it is smaller than every tetrahedron.*
7. *Only dodecahedra are larger than everything else.*
8. *All objects with nothing in front of them are tetrahedra.*
9. *Nothing is between two objects which are the same shape.*
10. *Nothing but a cube is between two other objects.*
11. ***b** has something behind it which has at least two objects behind it.*
12. *More than one thing is smaller than something larger than **b**.*

Submit your sentence file.

11.21 Using the symbols introduced in Table 1.2, page 30, translate the following into FOL. Do not introduce any additional names or predicates. Comment on any shortcomings in your translations. When you are done, submit your sentence file and turn in your comments to your instructor.

1. *Every student gave a pet to some other student sometime or other.*
2. *Claire is not a student unless she owned a pet (at some time or other).*
3. *No one ever owned both Folly and Scruffy at the same time.*
4. *No student fed every pet.*
5. *No one who owned a pet at 2:00 was angry.*
6. *No one gave Claire a pet this morning.* (Assume that "this morning" simply means before 12:00.)
7. *If Max ever gave Claire a pet, she owned it then and he didn't.*
8. *You can't give someone something you don't own.*
9. *Max fed all of his pets before Claire fed any of her pets.* (Assume that "Max's pets" are the pets he owned at 2:00, and the same for Claire.)
10. *Max gave Claire a pet between 2:00 and 3:00. It was hungry.*

11.22 ✎ Using the symbols introduced in Table 1.2, page 30, translate the following into colloquial English. Assume that each of the sentences is asserted at 2 p.m. on January 2, 2011, and use this fact to make your translations more natural. For example, you could translate $\mathsf{Owned(max, folly, 2{:}00)}$ as *Max owns Folly.*

1. $\forall x\,[\mathsf{Student}(x) \rightarrow \exists z\,(\mathsf{Pet}(z) \wedge \mathsf{Owned}(x, z, 2{:}00))]$
2. $\exists x\,[\mathsf{Student}(x) \wedge \forall z\,(\mathsf{Pet}(z) \rightarrow \mathsf{Owned}(x, z, 2{:}00))]$
3. $\forall x\,\forall t\,[\mathsf{Gave}(\mathsf{max}, x, \mathsf{claire}, t) \rightarrow \exists y\,\exists t'\,\mathsf{Gave}(\mathsf{claire}, x, y, t')]$
4. $\exists x\,[\mathsf{Owned}(\mathsf{claire}, x, 2{:}00) \wedge \exists t\,(t < 2{:}00 \wedge \mathsf{Gave}(\mathsf{max}, x, \mathsf{claire}, t))]$
5. $\exists x\,\exists t\,(1{:}55 < t \wedge t < 2{:}00 \wedge \mathsf{Gave}(\mathsf{max}, x, \mathsf{claire}, t))$
6. $\forall y\,[\mathsf{Person}(y) \rightarrow \exists x\,\exists t\,(1{:}55 < t \wedge t < 2{:}00 \wedge \mathsf{Gave}(\mathsf{max}, x, y, t))]$
7. $\exists z\,\{\mathsf{Student}(z) \wedge \forall y\,[\mathsf{Person}(y) \rightarrow \exists x\,\exists t\,(1{:}55 < t \wedge t < 2{:}00 \wedge$
   $\mathsf{Gave}(z, x, y, t))]\}$

11.23 ✎★ Translate the following into FOL. As usual, explain the meanings of the names, predicates, and function symbols you use, and comment on any shortcomings in your translations.

1. *There's a sucker born every minute.*
2. *Whither thou goest, I will go.*
3. *Soothsayers make a better living in the world than truthsayers.*
4. *To whom nothing is given, nothing can be required.*
5. *If you always do right, you will gratify some people and astonish the rest.*

SECTION 11.5

Ambiguity and context sensitivity

There are a couple of things that make the task of translating between English and first-order logic difficult. One is the sparseness of primitive concepts in FOL. While this sparseness makes the language easy to learn, it also means that there are frequently no very natural ways of saying what you want to say. You have to try to find circumlocutions available with the resources at hand. While this is often possible in mathematical discourse, it is frequently impossible for ordinary English. (We will return to this matter later.)

ambiguity

The other thing that makes it difficult is that English is rife with ambiguities, whereas the expressions of first-order logic are unambiguous (at least if the predicates used are unambiguous). Thus, confronted with a sentence of English, we often have to choose one among many possible interpretations in deciding on an appropriate translation. Just which is appropriate usually depends on context.

The ambiguities become especially vexing with quantified noun phrases. Consider, for example, the following joke, taken from *Saturday Night Live*:

Exercises

11.24 If you skipped the **You try it** section, go back and do it now. Save your sentence file as Sentences Max 1.

11.25 (Translating extended discourse)

- Open Reichenbach's World 1 and examine it. Check to see that all of the sentences in the following discourse are true in this world.

  > *There are (at least) two cubes. There is something between them. It is a medium dodecahedron. It is in front of a large dodecahedron. These two are left of a small dodecahedron. There are two tetrahedra.*

  Translate this discourse into a single first-order sentence. Check to see that your translation is true. Now check to see that your translation is false in Reichenbach's World 2.

- Open Reichenbach's World 2. Check to see that all of the sentences in the following discourse are true in this world.

  > *There are two tetrahedra. There is something between them. It is a medium dodecahedron. It is in front of a large dodecahedron. There are two cubes. These two are left of a small dodecahedron.*

  Translate this into a single first-order sentence. Check to see that your translation is true. Now check to see that your translation is false in Reichenbach's World 1. However, note that the English sentences in the two discourses are in fact exactly the same; they have just been rearranged! The moral of this exercise is that the correct translation of a sentence into first-order logic (or any other language) can be very dependent on context. Submit your sentence file.

11.26 (Ambiguity) Use Tarski's World to create a new sentence file and use it to translate the following sentences into FOL. Each of these sentences is ambiguous, so you should have two different translations of each. Put the two translations of sentence 1 in slots 1 and 2, the two translations of sentence 3 in slots 3 and 4, and so forth.

1. *Every cube is between a pair of dodecahedra.*
3. *Every cube to the right of a dodecahedron is smaller than it is.*
5. *Cube **a** is not larger than every dodecahedron.*

7. *No cube is to the left of some dodecahedron.*
9. *(At least) two cubes are between (at least) two dodecahedra.*

Now open Carroll's World. Which of your sentences are true in this world? You should find that exactly one translation of each sentence is true. If not, you should correct one or both of your translations. Notice that if you had had the world in front of you when you did the translations, it would have been harder to see the ambiguity in the English sentences. The world would have provided a context that made one interpretation the natural one. Submit your sentence file.

(Ambiguity and inference) Whether or not an argument is valid often hinges on how some ambiguous claim is taken. Here are two arguments, each of whose first premise is ambiguous. Translate each argument into FOL *twice, corresponding to the ambiguity in the first premise. (In 11.27, ignore the reading where "someone" means "everyone.") Under one translation the conclusion follows: prove it. Under the other, it does not: describe a situation in which the premises are true but the conclusion false.*

11.27 ✎

| Everyone admires someone who has red hair.
| Anyone who admires himself is conceited.
|---
| Someone with red hair is conceited.

11.28 ✎

| All that glitters is not gold.
| This ring glitters.
|---
| This ring is not gold.

Section 11.6

Translations using function symbols

relations and functions

Intuitively, functions are a kind of relation. One's mother is one's mother because of a certain relationship you and she bear to one another. Similarly, $2 + 3 = 5$ because of a certain relationship between two, three, and five. Building on this intuition, it is not hard to see that anything that can be expressed in FOL with function symbols can also be expressed in a version of FOL where the function symbols have been replaced by relation symbols.

The basic idea can be illustrated easily. Let us use mother as a unary function symbol, but MotherOf as a *binary* relation symbol. Thus, for example, mother(max) = nancy and MotherOf(nancy, max) both state that Nancy is the mother of Max.

The basic claim is that anything we can say with the function symbol we can say in some other way using the relation symbol. As an example, here is a simple sentence using the function symbol:

$$\forall x\, \mathsf{OlderThan}(\mathsf{mother}(x), x)$$

It expresses the claim that a person's mother is always older than the person. To express the same thing with the relation symbol, we might write

$$\forall x\, \exists y\, [\mathsf{MotherOf}(y, x) \land \mathsf{OlderThan}(y, x)]$$

Actually, one might wonder whether the second sentence quite manages to express the claim made by the first, since all it says is that everyone has at least one mother who is older than they are. One might prefer something like

$$\forall x\, \forall y\, [\mathsf{MotherOf}(y, x) \to \mathsf{OlderThan}(y, x)]$$

This says that every mother of everyone is older than they are. But this too seems somewhat deficient. A still better translation would be to conjoin one of the above sentences with the following two sentences which, together, assert that the relation of being the mother of someone is functional. Everyone has at least one, and everyone has at most one.

$$\forall x\, \exists y\, \mathsf{MotherOf}(y, x)$$

and

$$\forall x\, \forall y\, \forall z\, [(\mathsf{MotherOf}(y, x) \land \mathsf{MotherOf}(z, x)) \to y = z]$$

We will study this sort of thing much more in Chapter 14, where we will see that these two sentences can jointly be expressed by one rather opaque sentence:

$$\forall x\, \exists y\, [\mathsf{MotherOf}(y, x) \land \forall z\, [\mathsf{MotherOf}(z, x) \to y = z]]$$

And, if we wanted to, we could then incorporate our earlier sentence and express the first claim by means of the horrendous looking:

$$\forall x\, \exists y\, [\mathsf{MotherOf}(y, x) \land \mathsf{OlderThan}(y, x) \land \forall z\, [\mathsf{MotherOf}(z, x) \to y = z]]$$

By now it should be clearer why function symbols are so useful. Look at all the connectives and additional quantifiers that have come into translating our very simple sentence

$$\forall x\, \mathsf{OlderThan}(\mathsf{mother}(x), x)$$

We present some exercises below that will give you practice translating sentences from English into FOL, sentences that show why it is nice to have function symbols around.

Remember

Anything you can express using an n-ary function symbol can also be expressed using an $n+1$-ary relation symbol, plus the identity predicate, but at a cost in terms of the complexity of the sentences used.

Exercises

11.29 ✎ Translate the following sentences into FOL twice, once using the function symbol mother, once using the relation symbol MotherOf.

1. *Claire's mother is older than Max's mother.*
2. *Everyone's mother's mother is older than Melanie.*
3. *Someone's mother's mother is younger than Mary.*

11.30 ✎ Translate the following into a version of FOL that has function symbols height, mother, and father, the predicate >, and names for the people mentioned.

1. *Mary's father is taller than Mary but not taller than Claire's father.*
2. *Someone is taller than Claire's father.*
3. *Someone's mother is taller than their father.*
4. *Everyone is taller than someone else.*
5. *No one is taller than himself.*
6. *Everyone but J.R. who is taller than Claire is taller than J.R.*
7. *Everyone who is shorter than Claire is shorter than someone who is shorter than Melanie's father.*
8. *Someone is taller than Jon's paternal grandmother but shorter than his maternal grandfather.*

Say which sentences are true, referring to the table in Figure 9.1 (p. 256). Take the domain of quantification to be the people mentioned in the table. Turn in your answers.

11.31 ✎ Translate the following sentences into the blocks language augmented with the four function symbols lm, rm, fm, and bm discussed in Section 1.5 (page 33) and further discussed in connection with quantifiers in Section 9.7 (page 254). Tell which of these sentences are true in Malcev's World.

1. *Every cube is to the right of the leftmost block in the same row.*
2. *Every block is in the same row as the leftmost block in the same row.*
3. *Some block is in the same row as the backmost block in the same column.*
4. *Given any two blocks, the first is the leftmost block in the same row as the second if and only if there is nothing to the left of the second.*
5. *Given any two blocks, the first is the leftmost block in the same row as the second if and only if there is nothing to the left of the second and the the two blocks are in the same row.*

Turn in your answers.

11.32 ✎ Using the first-order language of arithmetic described earlier, express each of the following in FOL.

1. Every number is either 0 or greater than 0.
2. The sum of any two numbers greater than 1 is smaller than the product of the same two numbers.
3. Every number is even. [This is false, of course.]
4. If $x^2 = 1$ then $x = 1$. [Hint: Don't forget the implicit quantifier.]
5. ⋆⋆ For any number x, if $ax^2+bx+c = 0$ then either $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ or $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$. In this problem treat a, b, c as constants but x as a variable, as usual in algebra.

Section 11.7

Prenex form

When we translate complex sentences of English into FOL, it is common to end up with sentences where the quantifiers and connectives are all scrambled together. This is usually due to the way in which the translations of complex noun phrases of English use both quantifiers and connectives:

$$\forall x\, (P(x) \to \ldots)$$

$$\exists x\, (P(x) \land \ldots)$$

As a result, the translation of (the most likely reading of) a sentence like *Every cube to the left of a tetrahedron is in back of a dodecahedron* ends up looking like

$$\forall x\, [(\mathsf{Cube}(x) \land \exists y\, (\mathsf{Tet}(y) \land \mathsf{LeftOf}(x, y))) \to \exists y\, (\mathsf{Dodec}(y) \land \mathsf{BackOf}(x, y))]$$

While this is the most natural translation of our sentence, there are situations where it is not the most convenient one. It is sometimes important that we be able to rearrange sentences like this so that all the quantifiers are out in front and all the connectives in back. Such a sentence is said to be in *prenex form*, since all the quantifiers come first.

prenex form

Stated more precisely, a wff is in *prenex normal form* if either it contains no quantifiers at all, or else is of the form

$$Q_1 v_1 Q_2 v_2 \ldots Q_n v_n P$$

where each Q_i is either $\forall$ or $\exists$, each v_i is some variable, and the wff P is quantifier-free.

Exercises

Derive the following from the principles given earlier, by replacing $\rightarrow$ by its definition in terms of $\lor$ and $\neg$.

11.33 ✎ $\forall x\, P \rightarrow Q \quad \Leftrightarrow \quad \exists x\, [P \rightarrow Q]$ if x not free in Q

11.34 ✎ $\exists x\, P \rightarrow Q \quad \Leftrightarrow \quad \forall x\, [P \rightarrow Q]$ if x not free in Q

11.35 ✎ $P \rightarrow \forall x\, Q \quad \Leftrightarrow \quad \forall x\, [P \rightarrow Q]$ if x not free in P

11.36 ✎ $P \rightarrow \exists x\, Q \quad \Leftrightarrow \quad \exists x\, [P \rightarrow Q]$ if x not free in P

11.37 ➶ (Putting sentences in Prenex form) Open Jon Russell's Sentences. You will find ten sentences, at the odd numbered positions. Write a prenex form of each sentence in the space below it. Save your sentences. Open a few worlds, and make sure that your prenex form has the same truth value as the sentence above it.

11.38 ➶ (Some invalid quantifier manipulations) We remarked above on the invalidity of some quantifier manipulations that are superficially similar to the valid ones. In fact, in both cases one side is a logical consequence of the other side, but not vice versa. We will illustrate this. Build a world in which (1) and (3) below are true, but (2) and (4) are false.

1. $\forall x\, [\mathsf{Cube}(x) \lor \mathsf{Tet}(x)]$
2. $\forall x\, \mathsf{Cube}(x) \lor \forall x\, \mathsf{Tet}(x)$
3. $\exists x\, \mathsf{Cube}(x) \land \exists x\, \mathsf{Small}(x)$
4. $\exists x\, [\mathsf{Cube}(x) \land \mathsf{Small}(x)]$

Section 11.8

Some extra translation problems

Some instructors concentrate more on translation than others. For those who like to emphasize this skill, we present some additional challenging exercises here.

Exercises

11.39 (Translation) Open Peirce's World. Look at it in 2-D to remind yourself of the hidden objects. Start a new sentence file where you will translate the following English sentences. Again, be sure to check each of your translations to see that it is indeed a true sentence.

1. *Everything is either a cube or a tetrahedron.*
2. *Every cube is to the left of every tetrahedron.*
3. *There are at least three tetrahedra.*
4. *Every small cube is in back of a particular large cube.*
5. *Every tetrahedron is small.*
6. *Every dodecahedron is smaller than some tetrahedron.* [Note: This is vacuously true in this world.]

Now let's change the world so that none of the English sentences are true. (We can do this by changing the large cube in front to a dodecahedron, the large cube in back to a tetrahedron, and deleting the two small tetrahedra in the far right column.) If your answers to 1–5 are correct, all of your translations should be false as well. If not, you have made a mistake in translation. Make further changes, and check to see that the truth values of your translations track those of the English sentences. Submit your sentence file.

11.40 ★★ (More translations for practice) This exercise is just to give you more practice translating sentences of various sorts. They are all true in Skolem's World, in case you want to look while translating.

- Translate the following sentences.

1. *Not every cube is smaller than every tetrahedron.*
2. *No cube is to the right of anything.*
3. *There is a dodecahedron unless there are at least two large objects.*
4. *No cube with nothing in back of it is smaller than another cube.*
5. *If any dodecahedra are small, then they are between two cubes.*
6. *If a cube is medium or is in back of something medium, then it has nothing to its right except for tetrahedra.*
7. *The further back a thing is, the larger it is.*
8. *Everything is the same size as something else.*
9. *Every cube has a tetrahedron of the same size to its right.*
10. *Nothing is the same size as two (or more) other things.*
11. *Nothing is between objects of shapes other than its own.*

- Open Skolem's World. Notice that all of the above English sentences are true. Verify that the same holds of your translations.
- This time, rather than open other worlds, make changes to Skolem's World and see that the truth value of your translations track that of the English sentence. For example, consider sentence 5. Add a small dodecahedron between the front two cubes. The English sentence is still true. Is your translation? Now move the dodecahedron over between two tetrahedra. The English sentence is false. Is your translation? Now make the dodecahedron medium. The English sentence is again true. How about your translation?

Submit your sentence file.

11.41 ✎⋆ Using the symbols introduced in Table 1.2, page 30, translate the following into FOL. Do not introduce any additional names or predicates. Comment on any shortcomings in your translations.

1. *No student owned two pets at a time.*
2. *No student owned two pets until Claire did.*
3. *Anyone who owns a pet feeds it sometime.*
4. *Anyone who owns a pet feeds it sometime while they own it.*
5. *Only pets that are hungry are fed.*

11.42 ✎⋆ Translate the following into FOL. As usual, explain the meanings of the names, predicates, and function symbols you use, and comment on any shortcomings in your translations.

1. *You should always except the present company.*
2. *There was a jolly miller once*
   Lived on the River Dee;
   He worked and sang from morn till night
   No lark more blithe than he.
3. *Man is the only animal that blushes. Or needs to.*
4. *You can fool all of the people some of the time, and some of the people all of the time, but you can't fool all of the people all of the time.*
5. *Everybody loves a lover.*

11.43 ✎⋆ Give two translations of each of the following and discuss which is the most plausible reading, and why.

1. *Every senior in the class likes his or her computer, and so does the professor.* [Treat "the professor" as a name here and in the next sentence.]
2. *Every senior in the class likes his or her advisor, and so does the professor.*
3. *In some countries, every student must take an exam before going to college.*
4. *In some countries, every student learns a foreign language before going to college.*

11.44 ✎ (Using DeMorgan's Laws in mathematics) The DeMorgan Laws for quantifiers are quite helpful in mathematics. A function f on real numbers is said to be continuous at 0 if, intuitively, $f(x)$ can be kept close to $f(0)$ by keeping x close enough to 0. If you have had calculus then you will probably recognize the following a way to make this definition precise:

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \, (| \, x \, | < \delta \rightarrow | \, f(x) - f(0) \, | < \epsilon)$$

Here "$\forall \epsilon > 0(\ldots)$" is shorthand for "$\forall \epsilon(\epsilon > 0 \rightarrow \ldots)$". Similarly, "$\exists \delta > 0(\ldots)$" is shorthand for "$\exists \delta(\delta > 0 \wedge \ldots)$". Use DeMorgan's Laws to express the claim that f is not continuous at 0 in prenex form. You may use the same kind of shorthand we have used. Turn in your solution.

11.45 ✎⋆ Translate the following two sentences into FOL:

1. *If everyone comes to the party, I will have to buy more food.*
2. *There is someone such that if that person comes to the party, I will have to buy more food.*

The natural translations of these turn out to have forms that are equivalent, according to the equivalence in Problem 11.33. But clearly the English sentences do not mean the same thing. Explain what is going on here. Are the natural translations really correct?

We began the discussion of the logic of quantified sentences in Chapter 10 by looking at the following arguments:

1. $\forall x\,(\mathsf{Cube}(x) \rightarrow \mathsf{Small}(x))$
   $\forall x\,\mathsf{Cube}(x)$

   $\forall x\,\mathsf{Small}(x)$

2. $\forall x\,\mathsf{Cube}(x)$
   $\forall x\,\mathsf{Small}(x)$

   $\forall x\,(\mathsf{Cube}(x) \land \mathsf{Small}(x))$

We saw there that the truth functional rules did not suffice to establish these arguments. In this chapter we have seen (on page 335) how to establish the first using valid methods that apply to the quantifiers. Let's conclude this discussion by giving an informal proof of the second.

Proof: Let *d* be any object in the domain of discourse. By the first premise, we obtain (by universal elimination) $\mathsf{Cube}(d)$. By the second premise, we obtain $\mathsf{Small}(d)$. Hence we have $(\mathsf{Cube}(d) \land \mathsf{Small}(d))$. But since *d* is an arbitrary object in the domain, we can conclude $\forall x\,(\mathsf{Cube}(x) \land \mathsf{Small}(x))$, by universal generalization.

Exercises

The following exercises each contain a formal argument and something that purports to be an informal proof of it. Some of these proofs are correct while others are not. Give a logical critique of the purported proof. Your critique should take the form of a short essay that makes explicit each proof step or method of proof used, indicating whether it is valid or not. If there is a mistake, see if can you patch it up by giving a correct proof of the conclusion from the premises. If the argument in question is valid, you should be able to fix up the proof. If the argument is invalid, then of course you will not be able to fix the proof.

12.1 ✎

$\forall x\,[(\mathsf{Brillig}(x) \lor \mathsf{Tove}(x)) \rightarrow (\mathsf{Mimsy}(x) \land \mathsf{Gyre}(x))]$
$\forall y\,[(\mathsf{Slithy}(y) \lor \mathsf{Mimsy}(y)) \rightarrow \mathsf{Tove}(y)]$
$\exists x\,\mathsf{Slithy}(x)$

$\exists x\,[\mathsf{Slithy}(x) \land \mathsf{Mimsy}(x)]$

Purported proof: By the third premise, we know that something in the domain of discourse is slithy. Let *b* be one of these slithy things. By the second premise, we know that *b* is a tove. By the first premise, we see that *b* is mimsy. Thus, *b* is both slithy and mimsy. Hence, something is both slithy and mimsy.

12.2 ✎

$\forall x\,[\mathsf{Brillig}(x) \rightarrow (\mathsf{Mimsy}(x) \wedge \mathsf{Slithy}(x))]$
$\forall y\,[(\mathsf{Slithy}(y) \vee \mathsf{Mimsy}(y)) \rightarrow \mathsf{Tove}(y)]$
$\forall x\,[\mathsf{Tove}(x) \rightarrow (\mathsf{Outgrabe}(x, b) \wedge \mathsf{Brillig}(x))]$

$\forall z\,[\mathsf{Brillig}(z) \leftrightarrow \mathsf{Mimsy}(z)]$

Purported proof: In order to prove the conclusion, it suffices to prove the logically equivalent sentence obtained by conjoining the following two sentences:

(1) $\forall x\,[\mathsf{Brillig}(x) \rightarrow \mathsf{Mimsy}(x)]$

(2) $\forall x\,[\mathsf{Mimsy}(x) \rightarrow \mathsf{Brillig}(x)]$

We prove these by the method of general conditional proof, in turn. To prove (1), let *b* be anything that is brillig. Then by the first premise it is both mimsy and slithy. Hence it is mimsy, as desired. Thus we have established (1).

To prove (2), let *b* be anything that is mimsy. By the second premise, *b* is also tove. But then by the final premise, *b* is brillig, as desired. This concludes the proof.

12.3 ✎

$\forall x\,[(\mathsf{Brillig}(x) \wedge \mathsf{Tove}(x)) \rightarrow \mathsf{Mimsy}(x)]$
$\forall y\,[(\mathsf{Tove}(y) \vee \mathsf{Mimsy}(y)) \rightarrow \mathsf{Slithy}(y)]$
$\exists x\,\mathsf{Brillig}(x) \wedge \exists x\,\mathsf{Tove}(x)$

$\exists z\,\mathsf{Slithy}(z)$

Purported proof: By the third premise, we know that there are brillig toves. Let *b* be one of them. By the first premise, we know that *b* is mimsy. By the second premise, we know that *b* is slithy. Hence, there is something that is slithy.

The following exercises each contains an argument; some are valid, some not. If the argument is valid, give an informal proof. If it is not valid, use Tarski's World to construct a counterexample.

12.4 ➶|✎

$\forall y\,[\mathsf{Cube}(y) \vee \mathsf{Dodec}(y)]$
$\forall x\,[\mathsf{Cube}(x) \rightarrow \mathsf{Large}(x)]$
$\exists x\,\neg\mathsf{Large}(x)$

$\exists x\,\mathsf{Dodec}(x)$

12.5 ➶|✎

$\forall y\,[\mathsf{Cube}(y) \vee \mathsf{Dodec}(y)]$
$\forall x\,[\mathsf{Cube}(x) \rightarrow \mathsf{Large}(x)]$
$\exists x\,\neg\mathsf{Large}(x)$

$\exists x\,[\mathsf{Dodec}(x) \wedge \mathsf{Small}(x)]$

12.6 ➶|✎

$\forall x\,[\mathsf{Cube}(x) \vee \mathsf{Dodec}(x)]$
$\forall x\,[\neg\mathsf{Small}(x) \rightarrow \mathsf{Tet}(x)]$

$\neg\exists x\,\mathsf{Small}(x)$

12.7 ➶|✎

$\forall x\,[\mathsf{Cube}(x) \vee \mathsf{Dodec}(x)]$
$\forall x\,[\mathsf{Cube}(x) \rightarrow (\mathsf{Large}(x) \wedge \mathsf{LeftOf}(c, x))]$
$\forall x\,[\neg\mathsf{Small}(x) \rightarrow \mathsf{Tet}(x)]$

$\exists z\,\mathsf{Dodec}(z)$

12.8

$\forall x\,[\mathsf{Cube}(x) \lor (\mathsf{Tet}(x) \land \mathsf{Small}(x))]$
$\exists x\,[\mathsf{Large}(x) \land \mathsf{BackOf}(x, c)]$

$\exists x\,[\mathsf{FrontOf}(c, x) \land \mathsf{Cube}(x)]$

12.9

$\forall x\,[(\mathsf{Cube}(x) \land \mathsf{Large}(x)) \lor (\mathsf{Tet}(x) \land \mathsf{Small}(x))]$
$\forall x\,[\mathsf{Tet}(x) \to \mathsf{BackOf}(x, c)]$

$\forall x\,[\mathsf{Small}(x) \to \mathsf{BackOf}(x, c)]$

12.10

$\forall x\,[\mathsf{Cube}(x) \lor (\mathsf{Tet}(x) \land \mathsf{Small}(x))]$
$\exists x\,[\mathsf{Large}(x) \land \mathsf{BackOf}(x, c)]$

$\forall x\,[\mathsf{Small}(x) \to \neg\mathsf{BackOf}(x, c)]$

Section 12.4

Proofs involving mixed quantifiers

There are no new methods of proof that apply specifically to sentences with mixed quantifiers, but the introduction of mixed quantifiers forces us to be more explicit about some subtleties having to do with the interaction of methods that introduce new names into a proof: existential instantiation, general conditional proof, and universal generalization. It turns out that problems can arise from the interaction of these methods of proof.

Let us begin by illustrating the problem. Consider the following argument:

$\exists y\,[\mathsf{Girl}(y) \land \forall x\,(\mathsf{Boy}(x) \to \mathsf{Likes}(x, y))]$

$\forall x\,[\mathsf{Boy}(x) \to \exists y\,(\mathsf{Girl}(y) \land \mathsf{Likes}(x, y))]$

If the domain of discourse were the set of children in a kindergarten class, the conclusion would say every boy in the class likes some girl or other, while the premise would say that there is some girl who is liked by every boy. Since this is valid, let's start by giving a proof of it.

Proof: Assume the premise. Thus, at least one girl is liked by every boy. Let c be one of these popular girls. To prove the conclusion we will use general conditional proof. Assume that d is any boy in the class. We want to prove that d likes some girl. But every boy likes c, so d likes c. Thus d likes some girl, by existential generalization. Since d was an arbitrarily chosen boy, the conclusion follows.

> Fred does shave himself, then he doesn't, since by the assumption he does not shave any man of the town who shaves himself. So now assume the other possibility, namely, that Fred doesn't shave himself. But then since Fred shaves every man of the town who doesn't shave himself, he must shave himself. We have shown that a contradiction follows from each possibility. By proof by cases, then, we have established a contradiction from our original assumption. This contradiction shows that our assumption is wrong, so there is no such town.

The conflict between our intuition that there could be such a town on the one hand, and the proof that there can be none on the other hand, has caused this result to be known as the Barber Paradox.

Barber Paradox

Actually, though, there is a subtle sexist flaw in this proof. Did you spot it? It came in our use of the name "Fred." By naming the barber Fred, we implicitly assumed the barber was a man, an assumption that was needed to complete the proof. After all, it is only about *men* that we know the barber shaves those who do not shave themselves. Nothing is said about women, children, or other inhabitants of the town.

The proof, though flawed, is not worthless. What it really shows is that if there is a town with such a barber, then that barber is not a man of the town. The barber might be a woman, or maybe a man from some other town. In other words, the proof works fine to show that the following is a first-order validity:

$$\neg\exists z\,\exists x\,[\mathsf{ManOf}(x,z) \land \forall y\,(\mathsf{ManOf}(y,z) \to (\mathsf{Shave}(x,y) \leftrightarrow \neg\mathsf{Shave}(y,y)))]$$

There are many variations on this example that you can use to amaze, amuse, or annoy your family with when you go home for the holidays. We give a couple examples in the exercises (see Exercises 12.13 and 12.28).

Exercises

These exercises each contain a purported proof. If it is correct, say so. If it is incorrect, explain what goes wrong using the notions presented above.

12.11 ✎

| There is a number greater than every other number.

Purported proof: Let n be an arbitrary number. Then n is less than some other number, $n+1$ for example. Let m be any such number. Thus $n \leq m$. But n is an arbitrary number, so every number is less or equal m. Hence there is a number that is greater than every other number.

12.12 ✎

$\forall x\,[\mathsf{Person}(x) \rightarrow \exists y\,\forall z\,[\mathsf{Person}(z) \rightarrow \mathsf{GivesTo}(x,y,z)]]$

$\forall x\,[\mathsf{Person}(x) \rightarrow \forall z\,(\mathsf{Person}(z) \rightarrow \exists y\,\mathsf{GivesTo}(x,y,z))]$

Purported proof: Let us assume the premise and prove the conclusion. Let b be an arbitrary person in the domain of discourse. We need to prove

$$\forall z\,(\mathsf{Person}(z) \rightarrow \exists y\,\mathsf{GivesTo}(b,y,z))$$

Let c be an arbitrary person in the domain of discourse. We need to prove

$$\exists y\,\mathsf{GivesTo}(b,y,c)$$

But this follows directly from our premise, since there is something that b gives to everyone.

12.13 ✎

Harrison admires only great actors who do not admire themselves

Harrison admires all great actors who do not admire themselves.

Harrison is not a great actor.

Purported proof: Toward a proof by contradiction, suppose that Harrison is a great actor. Either Harrison admires himself or he doesn't. We will show that either case leads to a contradiction, so that our assumption that Harrison is a great actor must be wrong. First, assume that Harrison does admire himself. By the first premise and our assumption that Harrison is a great actor, Harrison does not admire himself, which is a contradiction. For the other case, assume that Harrison does not admire himself. But then by the second premise and our assumption that Harrison is a great actor, Harrison does admire himself after all. Thus, under either alternative, we have our contradiction.

12.14 ✎

There is at most one object.

Purported proof: Toward a proof by contradiction, suppose that there is more than one object in the domain of discourse. Let c be any one of these objects. Then there is some other object d, so that $d \neq c$. But since c was arbitrary, $\forall x\,(d \neq x)$. But then, by universal instantiation, $d \neq d$. But $d = d$, so we have our contradiction. Hence there can be at most one object in the domain of discourse.

12.15 ✎

$\forall x \forall y \forall z\, [(\mathsf{Outgrabe}(x,y) \land \mathsf{Outgrabe}(y,z)) \to \mathsf{Outgrabe}(x,z)]$
$\forall x \forall y\, [\mathsf{Outgrabe}(x,y) \to \mathsf{Outgrabe}(y,x)]$
$\exists x \exists y\, \mathsf{Outgrabe}(x,y)$

$\forall x\, \mathsf{Outgrabe}(x,x)$

Purported proof: Applying existential instantiation to the third premise, let b and c be arbitrary objects in the domain of discourse such that b outgrabes c. By the second premise, we also know that c outgrabes b. Applying the first premise (with $x = z = b$ and $y = c$ we see that b outgrabes itself. But b was arbitrary. Thus by universal generalization, $\forall x\, \mathsf{Outgrabe}(x,x)$.

The next three exercises contain arguments from a single set of premises. In each case decide whether or not the argument is valid. If it is, give an informal proof. If it isn't, use Tarski's World to construct a counterexample.

12.16 ➶|✎

$\forall x \forall y\, [\mathsf{LeftOf}(x,y) \to \mathsf{Larger}(x,y)]$
$\forall x\, [\mathsf{Cube}(x) \to \mathsf{Small}(x)]$
$\forall x\, [\mathsf{Tet}(x) \to \mathsf{Large}(x)]$
$\forall x \forall y\, [(\mathsf{Small}(x) \land \mathsf{Small}(y)) \to \neg\mathsf{Larger}(x,y)]$

$\neg\exists x \exists y\, [\mathsf{Cube}(x) \land \mathsf{Cube}(y) \land \mathsf{RightOf}(x,y)]$

12.17 ➶|✎

$\forall x \forall y\, [\mathsf{LeftOf}(x,y) \to \mathsf{Larger}(x,y)]$
$\forall x\, [\mathsf{Cube}(x) \to \mathsf{Small}(x)]$
$\forall x\, [\mathsf{Tet}(x) \to \mathsf{Large}(x)]$
$\forall x \forall y\, [(\mathsf{Small}(x) \land \mathsf{Small}(y)) \to \neg\mathsf{Larger}(x,y)]$

$\forall z\, [\mathsf{Medium}(z) \to \mathsf{Tet}(z)]$

12.18 ➶|✎

$\forall x \forall y\, [\mathsf{LeftOf}(x,y) \to \mathsf{Larger}(x,y)]$
$\forall x\, [\mathsf{Cube}(x) \to \mathsf{Small}(x)]$
$\forall x\, [\mathsf{Tet}(x) \to \mathsf{Large}(x)]$
$\forall x \forall y\, [(\mathsf{Small}(x) \land \mathsf{Small}(y)) \to \neg\mathsf{Larger}(x,y)]$

$\forall z \forall w\, [(\mathsf{Tet}(z) \land \mathsf{Cube}(w)) \to \mathsf{LeftOf}(z,w)]$

The next three exercises contain arguments from a single set of premises. In each, decide whether the argument is valid. If it is, give an informal proof. If it isn't valid, use Tarski's World to build a counterexample.

12.19

$\forall x\,[\mathsf{Cube}(x) \rightarrow \exists y\,\mathsf{LeftOf}(x,y)]$
$\neg\exists x\,\exists z\,[\mathsf{Cube}(x) \wedge \mathsf{Cube}(z) \wedge \mathsf{LeftOf}(x,z)]$
$\exists x\,\exists y\,[\mathsf{Cube}(x) \wedge \mathsf{Cube}(y) \wedge x \neq y]$

$\exists x\,\exists y\,\exists z\,[\mathsf{BackOf}(y,z) \wedge \mathsf{LeftOf}(x,z)]$

12.20

$\forall x\,[\mathsf{Cube}(x) \rightarrow \exists y\,\mathsf{LeftOf}(x,y)]$
$\neg\exists x\,\exists z\,[\mathsf{Cube}(x) \wedge \mathsf{Cube}(z) \wedge \mathsf{LeftOf}(x,z)]$
$\exists x\,\exists y\,[\mathsf{Cube}(x) \wedge \mathsf{Cube}(y) \wedge x \neq y]$

$\exists x\,\neg\mathsf{Cube}(x)$

12.21

$\forall x\,[\mathsf{Cube}(x) \rightarrow \exists y\,\mathsf{LeftOf}(x,y)]$
$\neg\exists x\,\exists z\,[\mathsf{Cube}(x) \wedge \mathsf{Cube}(z) \wedge \mathsf{LeftOf}(x,z)]$
$\exists x\,\exists y\,[\mathsf{Cube}(x) \wedge \mathsf{Cube}(y) \wedge x \neq y]$

$\exists x\,\exists y\,(x \neq y \wedge \neg\mathsf{Cube}(x) \wedge \neg\mathsf{Cube}(y))$

12.22 ★ Is the following logically true?

$$\exists x\,[\mathsf{Cube}(x) \rightarrow \forall y\,\mathsf{Cube}(y)]$$

If so, given an informal proof. If not, build a world where it is false.

12.23 Translate the following argument into FOL and determine whether or not the conclusion follows from the premises. If it does, give a proof.

Every child is either right-handed or intelligent.
No intelligent child eats liver.
There is a child who eats liver and onions.

There is a right-handed child who eats onions.

In the next three exercises, we work in the first-order language of arithmetic with the added predicates Even(x), Prime(x), *and* DivisibleBy(x, y), *where these have the obvious meanings (the last means that the natural number* y *divides the number* x *without remainder.) Prove the result stated in the exercise. In some cases, you have already done all the hard work in earlier problems.*

12.24 $\exists y\,[\mathsf{Prime}(y) \wedge \mathsf{Even}(y)]$

12.25 $\forall x\,[\mathsf{Even}(x) \leftrightarrow \mathsf{Even}(x^2)]$

12.26 $\forall x\,[\mathsf{DivisibleBy}(x^2, 3) \rightarrow \mathsf{DivisibleBy}(x^2, 9)]$

12.27 Are sentences (1) and (2) in Exercise 9.19 on page 252 logically equivalent? If so, give a proof. If not, explain why not.

12.28 Show that it would be impossible to construct a reference book that lists all and only those reference books that do not list themselves.

12.29 ✎* Call a natural number a *near prime* if its prime factorization contains at most two distinct primes. The first number which is not a near prime is $2 \times 3 \times 5 = 30$. Prove

$$\forall x\, \exists y\, [y > x \land \neg \mathsf{NearPrime}(y)]$$

You may appeal to our earlier result that there is no largest prime.

Section 12.5

Axiomatizing shape

Let's return to the project of giving axioms for the shape properties in Tarski's World. In Section 10.5, we gave axioms that described basic facts about the three shapes, but we stopped short of giving axioms for the binary relation SameShape. The reason we stopped was that the needed axioms require multiple quantifiers, which we had not covered at the time.

correctness of axioms

How do we choose which sentences to take as axioms? The main consideration is *correctness*: the axioms must be true in all relevant circumstances, either in virtue of the meanings of the predicates involved, or because we have restricted our attention to a specific type of circumstance.

The two possibilities are reflected in our first four axioms about shape, which we repeat here for ease of reference:

Basic Shape Axioms:

1. $\neg\exists x\,(\mathsf{Cube}(x) \land \mathsf{Tet}(x))$
2. $\neg\exists x\,(\mathsf{Tet}(x) \land \mathsf{Dodec}(x))$
3. $\neg\exists x\,(\mathsf{Dodec}(x) \land \mathsf{Cube}(x))$
4. $\forall x\,(\mathsf{Tet}(x) \lor \mathsf{Dodec}(x) \lor \mathsf{Cube}(x))$

The first three of these are correct in virtue of the meanings of the predicates; the fourth expresses a truth about all worlds of the sort that can be built in Tarski's World.

completeness of axioms

Of second importance, just behind correctness, is *completeness.* We say that a set of axioms is complete if, whenever an argument is intuitively valid (given the meanings of the predicates and the intended range of circumstances), its conclusion is a first-order consequence of its premises taken together with the axioms in question.

The notion of completeness, like that of correctness, is not precise, depending as it does on the vague notions of meaning and "intended circumstances."

Exercises

Give informal proofs of the following arguments, if they are valid, making use of any of the ten shape axioms as needed, so that your proof uses only first-order methods of proof. Be very explicit about which axioms you are using at various steps. If the argument is not valid, use Tarski's World to provide a counterexample.

12.30 ➶|✎

$$\begin{array}{|l} \exists x\, (\neg \mathsf{Cube}(x) \land \neg \mathsf{Dodec}(x)) \\ \exists x\, \forall y\, \mathsf{SameShape}(x, y) \\ \hline \forall x\, \mathsf{Tet}(x) \end{array}$$

12.31 ➶|✎

$$\begin{array}{|l} \forall x\, (\mathsf{Cube}(x) \to \mathsf{SameShape}(x, c)) \\ \hline \mathsf{Cube}(c) \end{array}$$

12.32 ➶|✎

$$\begin{array}{|l} \forall x\, \mathsf{Cube}(x) \lor \forall x\, \mathsf{Tet}(x) \lor \forall x\, \mathsf{Dodec}(x) \\ \hline \forall x\, \forall y\, \mathsf{SameShape}(x, y) \end{array}$$

12.33 ➶|✎

$$\begin{array}{|l} \forall x\, \forall y\, \mathsf{SameShape}(x, y) \\ \hline \forall x\, \mathsf{Cube}(x) \lor \forall x\, \mathsf{Tet}(x) \lor \forall x\, \mathsf{Dodec}(x) \end{array}$$

12.34 ➶|✎

$$\begin{array}{|l} \mathsf{SameShape}(b, c) \\ \hline \mathsf{SameShape}(c, b) \end{array}$$

12.35 ➶|✎

$$\begin{array}{|l} \mathsf{SameShape}(b, c) \\ \mathsf{SameShape}(c, d) \\ \hline \mathsf{SameShape}(b, d) \end{array}$$

12.36 ✎⋆ The last six shape axioms are quite intuitive and easy to remember, but we could have gotten by with fewer. In fact, there is a single sentence that completely captures the meaning of SameShape, given the first four axioms. This is the sentence that says that two things are the same shape if and only if they are both cubes, both tetrahedra, or both dodecahedra:

$$\begin{array}{lcl} \forall x\, \forall y\, (\mathsf{SameShape}(x, y) & \leftrightarrow & ((\mathsf{Cube}(x) \land \mathsf{Cube}(y)) \\ & & \lor(\mathsf{Tet}(x) \land \mathsf{Tet}(y)) \\ & & \lor(\mathsf{Dodec}(x) \land \mathsf{Dodec}(y))) \end{array}$$

Use this axiom and and the basic shape axioms (1)-(4) to give informal proofs of axioms (5) and (8).

12.37 ✎⋆ Let us imagine adding as new atomic sentences involving a binary predicate MoreSides. We assume that MoreSides(b, c) holds if block *b* has more sides than block *c*. See if you can come up with axioms that completely capture the meaning of this predicate. The natural way to do this involves two or three introduction axioms and three or four elimination axioms. Turn in your axioms to your instructor.

12.38 ✎⋆⋆ Find first-order axioms for the six size predicates of the blocks language. [Hint: use the axiomatization of shape to guide you.]

Fitch has generous uses of both $\forall$ rules. $\forall$ **Elim** will allow you to remove several universal quantifiers from the front a sentence simultaneously. For example, if you have proven $\forall x\, \forall y\, \mathsf{SameCol}(x, y)$ you could infer $\mathsf{SameCol}(f, c)$ in one step in Fitch. If you want to use the default mechanism to generate this step, you can enter the substitutions ": $x > f : y > c$" before checking the step.

In a like manner, you can often prove a sentence starting with more than one universal quantifier by means of a single application of $\forall$ **Intro**. You do this by starting a subproof with the appropriate number of boxed constants. If you then prove a sentence containing these constants you may end the subproof and infer the result of universally quantifying each of these constants using $\forall$ **Intro**. The default mechanism allows you to specify the variables to be used in the generated sentence by indicating the desired substitutions, for example ": $a > z : b > w$" will generate $\forall z\, \forall w\, \mathsf{R}(w, z)$ when applied to $\mathsf{R}(b, a)$. Notice the order used to specify substitutions: for $\forall$ **Elim** it will always be ": variable > name," while for $\forall$ **Intro** it must be ": name > variable."

Add Support Steps can be used with the $\forall$ **Intro** rule. If the focus step contains a universally quantified formula then the support will be a subproof with a new constant at the assumption step. The last step of the subproof will contain the appropriate instance of the universal formula. With the $\forall$ **Elim** rule a single support step containing a universal formula will be added. The support formula will be a universal generalization of the formula at the focus step, with the first constant replaced by the variable of quantification.

Remember

The formal rule of $\forall$ **Intro** corresponds to the informal method of general conditional proof, including the special case of universal generalization.

Exercises

13.1 If you skipped the **You try it** sections, go back and do them now. Submit the files Proof Universal 1 and Proof Universal 2.

For each of the following arguments, decide whether or not it is valid. If it is, use Fitch to give a formal proof. If it isn't, use Tarski's World to give a counterexample. In this chapter you are free to use **Taut Con** *to justify proof steps involving only propositional connectives.*

13.2

$\forall x\,(\mathsf{Cube}(x) \leftrightarrow \mathsf{Small}(x))$
$\forall x\,\mathsf{Cube}(x)$

$\forall x\,\mathsf{Small}(x)$

13.3

$\forall x\,\mathsf{Cube}(x)$
$\forall x\,\mathsf{Small}(x)$

$\forall x\,(\mathsf{Cube}(x) \land \mathsf{Small}(x))$

13.4

$\neg\forall x\,\mathsf{Cube}(x)$

$\neg\forall x\,(\mathsf{Cube}(x) \land \mathsf{Small}(x))$

13.5

$\forall x\,\forall y\,((\mathsf{Cube}(x) \land \mathsf{Dodec}(y)) \rightarrow \mathsf{Larger}(y, x))$
$\forall x\,\forall y\,(\mathsf{Larger}(x, y) \leftrightarrow \mathsf{LeftOf}(x, y))$

$\forall x\,\forall y\,((\mathsf{Cube}(x) \land \mathsf{Dodec}(y)) \rightarrow \mathsf{LeftOf}(y, x))$

13.6

$\forall x\,((\mathsf{Cube}(x) \land \mathsf{Large}(x)) \lor (\mathsf{Tet}(x) \land \mathsf{Small}(x)))$
$\forall x\,(\mathsf{Tet}(x) \rightarrow \mathsf{BackOf}(x, c))$
$\forall x\,\neg(\mathsf{Small}(x) \land \mathsf{Large}(x))$

$\forall x\,(\mathsf{Small}(x) \rightarrow \mathsf{BackOf}(x, c))$

(See Exercise 12.9. Notice that we have included a logical truth as an additional premise here.)

13.7

$\forall x\,\forall y\,((\mathsf{Cube}(x) \land \mathsf{Dodec}(y)) \rightarrow \mathsf{FrontOf}(x, y))$

$\forall x\,(\mathsf{Cube}(x) \rightarrow \forall y\,(\mathsf{Dodec}(y) \rightarrow \mathsf{FrontOf}(x, y)))$

13.8

$\forall x\,(\mathsf{Cube}(x) \rightarrow \forall y\,(\mathsf{Dodec}(y) \rightarrow \mathsf{FrontOf}(x, y)))$

$\forall x\,\forall y\,((\mathsf{Cube}(x) \land \mathsf{Dodec}(y)) \rightarrow \mathsf{FrontOf}(x, y))$

13.9 ★

$\forall x\,\forall y\,((\mathsf{Cube}(x) \land \mathsf{Dodec}(y)) \rightarrow \mathsf{Larger}(x, y))$
$\forall x\,\forall y\,((\mathsf{Dodec}(x) \land \mathsf{Tet}(y)) \rightarrow \mathsf{Larger}(x, y))$

$\forall x\,\forall y\,((\mathsf{Cube}(x) \land \mathsf{Tet}(y)) \rightarrow \mathsf{Larger}(x, y))$

Section 13.2

Existential quantifier rules

Recall that in our discussion of informal proofs, existential introduction was a simple proof step, whereas the elimination of ∃ was a subtle method of proof. Thus, in presenting our formal system, we begin with the introduction rule.

have been supplied by Fitch had we not specified this substitution? You could try it if you like.

3. When you are done, make sure you understand the completed proof. Save the file as Proof Existential 1. ◄

. . . . *Congratulations*

As with $\forall$, Fitch has generous uses of both $\exists$ rules. $\exists$ **Intro** will allow you to add several existential quantifiers to the front a sentence. For example, if you have proved $\mathsf{SameCol(b, a)}$ you could infer $\exists y\, \exists z\, \mathsf{SameCol(y, z)}$ in one step in Fitch. In a like manner, you can use a sentence beginning with more than one existential quantifier in a single application of $\exists$ **Elim**. You do this by starting a subproof with the appropriate number of boxed constants. If you then prove a sentence not containing these constants, you may end the subproof and infer the result using $\exists$ **Elim**.

The **Add Support Steps** command cannot be used with either of the $\exists$ rules.

Remember

The formal rule of $\exists$ **Elim** corresponds to the informal method of existential instantiation.

Exercises

13.10 If you skipped the **You try it** section, go back and do it now. Submit the file Proof Existential 1.

For each of the following arguments, decide whether or not it is valid. If it is, use Fitch to give a formal proof. If it isn't, use Tarski's World to give a counterexample. Remember that in this chapter you are free to use **Taut Con** *to justify proof steps involving only propositional connectives.*

13.11

| $\forall x\, (\mathsf{Cube}(x) \lor \mathsf{Tet}(x))$
| $\exists x\, \neg \mathsf{Cube}(x)$
|---
| $\exists x\, \neg \mathsf{Tet}(x)$

13.12

| $\forall x\, (\mathsf{Cube}(x) \lor \mathsf{Tet}(x))$
| $\exists x\, \neg \mathsf{Cube}(x)$
|---
| $\exists x\, \mathsf{Tet}(x)$

13.13

| $\forall y\, [\mathsf{Cube}(y) \lor \mathsf{Dodec}(y)]$
| $\forall x\, [\mathsf{Cube}(x) \to \mathsf{Large}(x)]$
| $\exists x\, \neg \mathsf{Large}(x)$
|---
| $\exists x\, \mathsf{Dodec}(x)$

13.14

| $\forall x\, (\mathsf{Cube}(x) \leftrightarrow \mathsf{Small}(x))$
| $\exists x\, \neg \mathsf{Cube}(x)$
|---
| $\exists x\, \neg \mathsf{Small}(x)$

13.15

| ∃x (Cube(x) → Small(x))
| ∀x Cube(x)
|—
| ∃x Small(x)

13.16

| ∃x ∃y Adjoins(x, y)
| ∀x ∀y (Adjoins(x, y)
| → ¬SameSize(x, y))
|—
| ∃x ∃y ¬SameSize(y, x)

In our discussion of the informal methods, we observed that the method that introduces new constants can interact to give defective proofs, if not used with care. The formal system $\mathcal{F}$ automatically prevents these misapplications of the quantifier rules. The next two exercises are designed to show you how the formal rules prevent these invalid steps by formalizing one of the fallacious informal proofs we gave earlier.

13.17 Here is a formalization of the pseudo-proof given on page 340:

1. ∀x ∃y SameCol(x, y)	
2. [c]	
3. ∃y SameCol(c, y)	∀ **Elim**: 1
4. [d] SameCol(c, d)	
5. SameCol(c, d)	**Reit**: 4
6. SameCol(c, d)	∃ **Elim**: 3, 4–5
7. ∀x SameCol(x, d)	∀ **Intro**: 2–6
8. ∃y ∀x SameCol(x, y)	∃ **Intro**: 7

1. Write this "proof" in a file using Fitch and check it out. You will discover that step 6 is incorrect; it violates the restriction on existential elimination that requires the constant d to appear only in the subproof where it is introduced. Notice that the other steps all check out, so if we *could* make that move, then the rest of the proof would be fine.
2. Construct a counterexample to the argument to show that no proof is possible.

Submit both files.

13.18 Let's contrast the faulty proof from the preceding exercise with a genuine proof that ∀x ∃y R(x, y) follows from ∃y ∀x R(x, y). Use Fitch to create the following proof.

$$
\begin{array}{lll}
 & 1.\ \exists y\, \forall x\, \mathsf{SameCol}(x, y) & \\
 & \quad 2.\ \boxed{d}\ \forall x\, \mathsf{SameCol}(x, d) & \\
 & \qquad 3.\ \boxed{c} & \\
 & \qquad 4.\ \mathsf{SameCol}(c, d) & \forall\ \textbf{Elim}: 2 \\
 & \qquad 5.\ \exists y\, \mathsf{SameCol}(c, y) & \exists\ \textbf{Intro}: 4 \\
 & \quad 6.\ \forall x\, \exists y\, \mathsf{SameCol}(x, y) & \forall\ \textbf{Intro}: 3\text{–}5 \\
 & 7.\ \forall x\, \exists y\, \mathsf{SameCol}(x, y) & \exists\ \textbf{Elim}: 1,\ 2\text{–}6
\end{array}
$$

Notice that in this proof, unlike the one in the previous exercise, both constant symbols c and d are properly sequestered within the subproofs where they are introduced. Therefore the quantifier rules have been applied properly. Submit your proof.

Section 13.3

Strategy and tactics

We have seen some rather simple examples of proofs using the new rules. In more interesting examples, however, the job of finding a proof can get pretty challenging. So a few words on how to approach these proofs will be helpful.

consider meaning

We have given you a general maxim and two strategies for finding sentential proofs. The maxim—to consider what the various sentences mean—is even more important with the quantifiers. Only if you remember what they mean and how the formal methods mirror common-sense informal methods will you be able to do any but the most boring of exercises.

informal proof as guide

Our first strategy was to try to come up with an informal proof of the goal sentence from the premises, and use it to try to figure out how your formal proof will proceed. This strategy, too, is even more important in proofs involving quantifiers, but it is a bit harder to apply. The key skill in applying the strategy is the ability to identify the formal rules implicit in your informal reasoning. This takes a bit of practice. Let's work through an example, to see some of the things you should be looking out for.

Suppose we want to prove that the following argument is valid:

$$
\begin{array}{l}
\exists x\, (\mathsf{Tet}(x) \land \mathsf{Small}(x)) \\
\forall x\, (\mathsf{Small}(x) \to \mathsf{LeftOf}(x, b)) \\
\hline
\exists x\, \mathsf{LeftOf}(x, b)
\end{array}
$$

4. Recall how we proved P(c). We said that if P(c) were not the case, then we would have ¬P(c), and hence ∃x ¬P(x). But this contradicted the assumption at step 2. Formalize this reasoning by filling in the rest of the proof. ◀

```
| 1. ¬∀x P(x)
|---
| | 2. ¬∃x ¬P(x)
| |---
| | | 3. [c]
| | |---
| | | | 4. ¬P(c)
| | | |---
| | | | 5. ∃x ¬P(x)          ∃ Intro: 4
| | | | 6. ⊥                 ⊥ Intro: 5, 2
| | | 7. ¬¬P(c)              ¬ Intro: 4–6
| | | 8. P(c)                ¬ Elim: 7
| | 9. ∀x P(x)               ∀ Intro: 3–8
| | 10. ⊥                    ⊥ Intro: 9, 1
| 11. ∃x ¬P(x)               ¬ Intro: 2–10
```

5. This completes our formal proof of ∃x ¬P(x) from the premise ¬∀x P(x). Verify your proof and save it as Proof Quantifier Strategy 1. ◀

. . . . *Congratulations*

Exercises

13.19 If you skipped the **You try it** section, go back and do it now. Submit the file Proof Quantifier Strategy 1.

Recall that in Exercises 12.1–12.3 on page 336, you were asked to give logical analyses of purported proofs of some arguments involving nonsense predicates. In the following exercises, we return to these arguments. If the argument is valid, submit a formal proof. If it is invalid, turn in an informal counterexample. If you submit a formal proof, be sure to use the Exercise file supplied with Fitch. In order to keep your hand in at using the propositional rules, we ask you not to use **Taut Con** *in these proofs.*

13.20

```
| ∀x [(Brillig(x) ∨ Tove(x)) → (Mimsy(x) ∧ Gyre(x))]
| ∀y [(Slithy(y) ∨ Mimsy(y)) → Tove(y)]
| ∃x Slithy(x)
|---
| ∃x [Slithy(x) ∧ Mimsy(x)]
```

(See Exercise 12.1 on p. 336.)

13.21

$\forall x\,[\mathsf{Brillig}(x) \rightarrow (\mathsf{Mimsy}(x) \land \mathsf{Slithy}(x))]$
$\forall y\,[(\mathsf{Slithy}(y) \lor \mathsf{Mimsy}(y)) \rightarrow \mathsf{Tove}(y)]$
$\forall x\,[\mathsf{Tove}(x) \rightarrow (\mathsf{Outgrabe}(x, b) \land \mathsf{Brillig}(x))]$

$\forall z\,[\mathsf{Brillig}(z) \leftrightarrow \mathsf{Mimsy}(z)]$

(See Exercise 12.2 on p. 337.)

13.22

$\forall x\,[(\mathsf{Brillig}(x) \land \mathsf{Tove}(x)) \rightarrow \mathsf{Mimsy}(x)]$
$\forall y\,[(\mathsf{Tove}(y) \lor \mathsf{Mimsy}(y)) \rightarrow \mathsf{Slithy}(y)]$
$\exists x\,\mathsf{Brillig}(x) \land \exists x\,\mathsf{Tove}(x)$

$\exists z\,\mathsf{Slithy}(z)$

(See Exercise 12.3, p. 337)

Some of the following arguments are valid, some are not. For each, either use Fitch to give a formal proof or use Tarski's World to construct a counterexample. In giving proofs, feel free to use **Taut Con** *if it helps.*

13.23

$\forall y\,[\mathsf{Cube}(y) \lor \mathsf{Dodec}(y)]$
$\forall x\,[\mathsf{Cube}(x) \rightarrow \mathsf{Large}(x)]$
$\exists x\,\neg\mathsf{Large}(x)$

$\exists x\,\mathsf{Dodec}(x)$

13.24

$\exists x\,(\mathsf{Cube}(x) \land \mathsf{Small}(x))$

$\exists x\,\mathsf{Cube}(x) \land \exists x\,\mathsf{Small}(x)$

13.25

$\exists x\,\mathsf{Cube}(x) \land \exists x\,\mathsf{Small}(x)$

$\exists x\,(\mathsf{Cube}(x) \land \mathsf{Small}(x))$

13.26

$\forall x\,(\mathsf{Cube}(x) \rightarrow \mathsf{Small}(x))$
$\forall x\,(\mathsf{Adjoins}(x, b) \rightarrow \mathsf{Small}(x))$

$\forall x\,((\mathsf{Cube}(x) \lor \mathsf{Small}(x)) \rightarrow \mathsf{Adjoins}(x, b))$

13.27

$\forall x\,(\mathsf{Cube}(x) \rightarrow \mathsf{Small}(x))$
$\forall x\,(\neg\mathsf{Adjoins}(x, b) \rightarrow \neg\mathsf{Small}(x))$

$\forall x\,((\mathsf{Cube}(x) \lor \mathsf{Small}(x)) \rightarrow \mathsf{Adjoins}(x, b))$

For each of the following, use Fitch to give a formal proof of the argument. These look simple but some of them are a bit tricky. Don't forget to first figure out an informal proof. Use **Taut Con** *whenever it is convenient but do not use* **FO Con**.

13.28

$\forall x\,\forall y\,\mathsf{Likes}(x, y)$

$\forall x\,\exists y\,\mathsf{Likes}(x, y)$

13.29

$\forall x\,(\mathsf{Small}(x) \rightarrow \mathsf{Cube}(x))$
$\exists x\,\neg\mathsf{Cube}(x) \rightarrow \exists x\,\mathsf{Small}(x)$

$\exists x\,\mathsf{Cube}(x)$

13.30

$\mathsf{Likes}(\mathsf{carl}, \mathsf{max})$
$\forall x\,[\exists y\,(\mathsf{Likes}(y, x) \lor \mathsf{Likes}(x, y)) \rightarrow \mathsf{Likes}(x, x)]$

$\exists x\,\mathsf{Likes}(x, \mathsf{carl})$

13.31

$\forall x\,\forall y\,[\mathsf{Likes}(x, y) \rightarrow \mathsf{Likes}(y, x)]$
$\exists x\,\forall y\,\mathsf{Likes}(x, y)$

$\forall x\,\exists y\,\mathsf{Likes}(x, y)$

The following valid arguments come in pairs. The validity of the first of the pair makes crucial use of the meanings of the blocks language predicates, whereas the second adds one or more premises, making the result a first-order valid argument. For the latter, give a proof that does not make use of **Ana Con**. *For the former, give a proof that uses* **Ana Con** *but only where the premises and conclusions of the citation are literals (including ⊥). You may use* **Taut Con** *but do not use* **FO Con** *in any of the proofs.*

13.32

$\neg\exists x\,(\mathsf{Tet}(x) \land \mathsf{Small}(x))$

$\forall x\,[\mathsf{Tet}(x) \to (\mathsf{Large}(x) \lor \mathsf{Medium}(x))]$

13.33

$\neg\exists x\,(\mathsf{Tet}(x) \land \mathsf{Small}(x))$
$\forall y\,(\mathsf{Small}(y) \lor \mathsf{Medium}(y) \lor \mathsf{Large}(y))$

$\forall x\,[\mathsf{Tet}(x) \to (\mathsf{Large}(x) \lor \mathsf{Medium}(x))]$

13.34

$\forall x\,(\mathsf{Dodec}(x) \to \mathsf{SameCol}(x, a))$
$\mathsf{SameCol}(a, c)$

$\forall x\,(\mathsf{Dodec}(x) \to \mathsf{SameCol}(x, c))$

13.35

$\forall x\,(\mathsf{Dodec}(x) \to \mathsf{SameCol}(x, a))$
$\mathsf{SameCol}(a, c)$
$\forall x\,\forall y\,\forall z\,((\mathsf{SameCol}(x, y) \land \mathsf{SameCol}(y, z)) \to \mathsf{SameCol}(x, z))$

$\forall x\,(\mathsf{Dodec}(x) \to \mathsf{SameCol}(x, c))$

13.36

$\forall x\,(\mathsf{Dodec}(x) \to \mathsf{LeftOf}(x, a))$
$\forall x\,(\mathsf{Tet}(x) \to \mathsf{RightOf}(x, a))$

$\forall x\,(\mathsf{SameCol}(x, a) \to \mathsf{Cube}(x))$

13.37

$\forall x\,(\mathsf{Dodec}(x) \to \mathsf{LeftOf}(x, a))$
$\forall x\,(\mathsf{Tet}(x) \to \mathsf{RightOf}(x, a))$
$\forall x\,\forall y\,(\mathsf{LeftOf}(x, y) \to \neg\mathsf{SameCol}(x, y))$
$\forall x\,\forall y\,(\mathsf{RightOf}(x, y) \to \neg\mathsf{SameCol}(x, y))$
$\forall x\,(\mathsf{Cube}(x) \lor \mathsf{Dodec}(x) \lor \mathsf{Tet}(x))$

$\forall x\,(\mathsf{SameCol}(x, a) \to \mathsf{Cube}(x))$

13.38

$\forall x\,(\mathsf{Cube}(x) \to \forall y\,(\mathsf{Dodec}(y) \to \mathsf{Larger}(x, y)))$
$\forall x\,(\mathsf{Dodec}(x) \to \forall y\,(\mathsf{Tet}(y) \to \mathsf{Larger}(x, y)))$
$\exists x\,\mathsf{Dodec}(x)$

$\forall x\,(\mathsf{Cube}(x) \to \forall y\,(\mathsf{Tet}(y) \to \mathsf{Larger}(x, y)))$

(Compare this with Exercise 13.9. The crucial difference is the presence of the third premise, not the difference in form of the first two premises.)

13.39

$\forall x\,(\mathsf{Cube}(x) \to \forall y\,(\mathsf{Dodec}(y) \to \mathsf{Larger}(x, y)))$
$\forall x\,(\mathsf{Dodec}(x) \to \forall y\,(\mathsf{Tet}(y) \to \mathsf{Larger}(x, y)))$
$\exists x\,\mathsf{Dodec}(x)$
$\forall x\,\forall y\,\forall z\,((\mathsf{Larger}(x, y) \land \mathsf{Larger}(y, z)) \to \mathsf{Larger}(x, z))$

$\forall x\,(\mathsf{Cube}(x) \to \forall y\,(\mathsf{Tet}(y) \to \mathsf{Larger}(x, y)))$

Section 13.4

Soundness and completeness

In Chapter 8 we raised the question of whether the deductive system $\mathcal{F}_T$ was sound and complete with respect to tautological consequence. The same issues arise with the full system $\mathcal{F}$, which contains the rules for the quantifiers and identity, in addition to the rules for the truth-functional connectives. Here, the target consequence relation is the notion of first-order consequence, rather than tautological consequence.

soundness of $\mathcal{F}$

The *soundness* question asks whether anything we can prove in $\mathcal{F}$ from premises $P_1, \ldots, P_n$ is indeed a first-order consequence of the premises. The *completeness* question asks the converse: whether every first-order consequence of a set of sentences can be proven from that set using the rules of $\mathcal{F}$.

completeness of $\mathcal{F}$

It turns out that both of these questions can be answered in the affirmative. Before actually proving this, however, we need to add more precision to the notion of first-order consequence, and this presupposes tools from set theory that we will introduce in Chapters 15 and 16. We state and prove the soundness theorem for first-order logic in Chapter 18. The completeness theorem for first-order logic is the main topic of Chapter 19.

Section 13.5

Some review exercises

In this section we present more problems to help you solidify your understanding of the methods of reasoning involving quantifiers. We also present some more interesting problems from a theoretical point of view.

Exercises

Some of the following arguments are valid, some are not. For each, either use Fitch to give a formal proof or use Tarski's World to construct a counterexample. In giving proofs, feel free to use **Taut Con** *if it helps.*

13.40

| ∃x Cube(x) ∧ Small(d)
|—
| ∃x (Cube(x) ∧ Small(d))

13.41

| ∀x (Cube(x) ∨ Small(x))
|—
| ∀x Cube(x) ∨ ∀x Small(x)

13.42 $\forall x\, \mathsf{Cube}(x) \lor \forall x\, \mathsf{Small}(x)$

$\forall x\, (\mathsf{Cube}(x) \lor \mathsf{Small}(x))$

Each of the following is a valid argument of a type discussed in Section 10.3. Use Fitch to give a proof of its validity. You may use **Taut Con** *freely in these proofs.*

13.43 $\neg\forall x\, \mathsf{Cube}(x)$

$\exists x\, \neg\mathsf{Cube}(x)$

13.44 $\neg\exists x\, \mathsf{Cube}(x)$

$\forall x\, \neg\mathsf{Cube}(x)$

13.45 $\forall x\, \neg\mathsf{Cube}(x)$

$\neg\exists x\, \mathsf{Cube}(x)$

13.46 (Change of bound variables)

$\forall x\, \mathsf{Cube}(x)$

$\forall y\, \mathsf{Cube}(y)$

13.47 (Change of bound variables)

$\exists x\, \mathsf{Tet}(x)$

$\exists y\, \mathsf{Tet}(y)$

13.48 (Null quantification)

$\mathsf{Cube}(b) \leftrightarrow \forall x\, \mathsf{Cube}(b)$

13.49 $\exists x\, P(x)$

$\forall x\, \forall y\, ((P(x) \land P(y)) \to x = y)$

$\exists x\, (P(x) \land \forall y\, (P(y) \to y = x))$

13.50 $\exists x\, (P(x) \land \forall y\, (P(y) \to y = x))$

$\forall x\, \forall y\, ((P(x) \land P(y)) \to x = y)$

13.51 ★★

$\exists x\, (P(x) \to \forall y\, P(y))$

[Hint: Review your answer to Exercise 12.22 where you should have given an informal proof of something of this form.]

13.52

$\neg\exists x\, \forall y\, [E(x, y) \leftrightarrow \neg E(y, y)]$

This result might be called Russell's Theorem. It is connected with the famous result known as Russell's Paradox, which is discussed in Section 15.9. In fact, it was upon discovering this that Russell invented the Barber Paradox, to explain his result to a general public.

13.53 Is $\exists x\, \exists y\, \neg\mathsf{LeftOf}(x, y)$ a first-order consequence of $\exists x\, \neg\mathsf{LeftOf}(x, x)$? If so, give a formal proof. If not, give a reinterpretation of LeftOf and an example where the premise is true and the conclusion is false.

The next exercises are intended to help you review the difference between first-order satisfiability and true logical possibility. All involve the four sentences in the file Padoa's Sentences. *Open that file now.*

13.54 ➶⋆ Any three of the sentences in Padoa's Sentences form a satisfiable set. There are four sets of three sentences, so to show this, build four worlds, World 13.54.123, World 13.54.124, World 13.54.134, and World 13.54.234, where the four sets are true. (Thus, for example, sentences 1, 2 and 4 should be true in World 13.54.124.)

13.55 ✎⋆ Give an informal proof that the four sentences in Padoa's Sentences taken together are inconsistent.

13.56 ✎⋆ Is the set of sentences in Padoa's Sentences first-order satisfiable, that is, satisfiable with some reinterpretation of the predicates other than identity? [Hint: Imagine a world where one of the blocks is a sphere.]

13.57 ✎⋆ Reinterpret the predicates Tet and Dodec in such a way that sentence 3 of Padoa's Sentences comes out true in World 13.54.124. Since this is the only sentence that uses these predicates, it follows that all four sentences would, with this reinterpretation, be true in this world. (This shows that the set is first-order satisfiable.)

13.58 ➶|✎⋆⋆ (Logical truth *versus* non-logical truth in all worlds) A distinction Tarski's World helps us to understand is the difference between sentences that are logically true and sentences that are, for reasons that have nothing to do with logic, true in all worlds. The notion of logical truth has to do with a sentence being true simply in virtue of the *meaning* of the sentence, and so no matter how the world is. However, some sentences are true in all worlds, not because of the meaning of the sentence or its parts, but because of, say, laws governing the world. We can think of the constraints imposed by the innards of Tarski's World as analogues of physical laws governing how the world can be. For example, the sentence which asserts that there are at most 12 objects happens to hold in all the worlds that we can construct with Tarski's World. However, it is not a logical truth.

Open Post's Sentences. Classify each sentence in one of the following ways: (A) a logical truth, (B) true in all worlds that can be depicted using Tarski's World, but not a logical truth, or (C) falsifiable in some world that can be depicted by Tarski's World. For each sentence of type (C), build a world in which it is false, and save it as World 13.58.x, where x is the number of the sentence. For each sentence of type (B), use a pencil and paper to depict a world in which it is false. (In doing this exercise, assume that Medium simply means *neither small nor large*, which seems plausible. However, it is not plausible to assume that Cube means *neither a dodecahedron nor tetrahedron*, so you should not assume anything like this.)

Similarly, to say *There are at most n cubes that are small*, we say *There are at most n things that are small cubes*. Finally, to say *There are exactly n cubes that are small*, we say *There are exactly n things that are small cubes*. These observations probably seem so obvious that they don't require mentioning. But we will soon see that nothing like this holds for some determiners, and that the consequences are rather important for the general theory of quantification.

Remember

The notations $\exists^{\geq n}$, $\exists^{\leq n}$, and $\exists^{!n}$ are abbreviations for complex FOL expressions meaning "there are at least/at most/exactly n things such that . . . ."

Exercises

14.1 If you skipped the **You try it** section, go back and do it now. Submit the files World Numerical 1 and World Numerical 2.

14.2 Give clear English translations of the following sentences of FOL. Which of the following are logically equivalent and which are not? Explain your answers.

1. $\exists!x\,\mathsf{Tove}(x)$ [Remember that the notation ∃! is an abbreviation, as explained above.]
2. $\exists x\,\forall y\,[\mathsf{Tove}(y) \rightarrow y = x]$
3. $\exists x\,\forall y\,[\mathsf{Tove}(y) \leftrightarrow y = x]$
4. $\forall x\,\forall y\,[(\mathsf{Tove}(x) \land \mathsf{Tove}(y)) \rightarrow x = y]$
5. $\forall x\,\forall y\,[(\mathsf{Tove}(x) \land \mathsf{Tove}(y)) \leftrightarrow x = y]$

14.3 ★ (Translating numerical claims) In this exercise we will try our hand at translating English sentences involving numerical claims.

- Using Tarski's World, translate the following English sentences.

  1. *There are at least two dodecahedra.*
  2. *There are at most two tetrahedra.*
  3. *There are exactly two cubes.*
  4. *There are only three things that are not small.*
  5. *There is a single large cube. No dodecahedron is in back of it.*

- Open Peano's World. Note that all of the English sentences are true in this world. Check to see that your translations are as well.

- Open Bolzano's World. Here sentences 1, 3, and 5 are the only true ones. Verify that your translations have the right truth values in this world.
- Open Skolem's World. Only sentence 5 is true in this world. Check your translations.
- Finally, open Montague's World. In this world, sentences 2, 3, and 5 are the only true ones. Check your translations.

14.4 (Saying more complicated things) Open Skolem's World. Create a file called Sentences 14.4 and describe the following features of Skolem's World.

1. Use your first sentence to say that there are only cubes and tetrahedra.
2. Next say that there are exactly three cubes.
3. Express the fact that every cube has a tetrahedron that is to its right but is neither in front of or in back of it.
4. Express the fact that at least one of the tetrahedra is between two other tetrahedra.
5. Notice that the further back something is, the larger it is. Say this.
6. Note that none of the cubes is to the right of any of the other cubes. Try to say this.
7. Observe that there is a single small tetrahedron and that it is in front of but to neither side of all the other tetrahedra. State this.

If you have expressed yourself correctly, there is very little you can do to Skolem's World without making at least one of your sentences false. Basically, all you can do is "stretch" things out, that is, move things apart while keeping them aligned. To see this, try making the following changes. (There's no need to turn in your answers, but try the changes.)

1. Add a new tetrahedron to the world. Find one of your sentences that comes out false. Move the new tetrahedron so that a different sentence comes out false.
2. Change the size of one of the objects. What sentence now comes out false?
3. Change the shape of one of the objects. What sentence comes out false?
4. Slide one of the cubes to the left. What sentence comes out false?
5. Rearrange the three cubes. What goes wrong now?

14.5 (Ambiguity and numerical quantification) In the **Try It** on page 314, we saw that the sentence

At least four medium dodecahedra are adjacent to a medium cube.

is ambiguous, having both a strong and a weak reading. Using Tarski's World, open a new sentence file and translate the strong and weak readings of this sentence into FOL as sentences (1) and (2). Remember that Tarski's World does not understand our abbreviation for "at least four" so you will need to write this out in full. Check that the first sentence is true in Anderson's First World but not in Anderson's Second World, while the second sentence is true in both worlds. Make some changes to the worlds to help you check that your translations express what you intend. Submit your sentence file.

14.6 ➶ (Games of incomplete information) As you recall, you can sometimes know that a sentence is true in a world without knowing how to play the game and win. Open Mostowski's World. Translate the following into first-order logic. Save your sentences as Sentences 14.6. Now, without using the 2-D view, make as good a guess as you can about whether the sentences are true or not in the world. Once you have assessed a given sentence, use **Verify** to see if you are right. Then, with the correct truth value checked, see how far you can go in playing the game. Quit whenever you get stuck, and play again. Can you predict in advance when you will be able to win? Do not look at the 2-D view until you have finished the whole exercise.

1. *There are at least two tetrahedra.*
2. *There are at least three tetrahedra.*
3. *There are at least two dodecahedra.*
4. *There are at least three dodecahedra.*
5. *Either there is a small tetrahedron behind a small cube or there isn't.*
6. *Every large cube is in front of something.*
7. *Every tetrahedron is in back of something.*
8. *Every small cube is in back of something.*
9. *Every cube has something behind it.*
10. *Every dodecahedron is small, medium, or large.*
11. *If* e *is to the left of every dodecahedron, then it is not a dodecahedron.*

Now modify the world so that the true sentences are still true, but so that it will be clear how to play the game and win. When you are done, just submit your sentence file.

14.7 ➶|✎ (Satisfiability) Recall that a set of sentences is satisfiable if there is world in which it is true. Determine whether the following set of sentences is satisfiable. If it is, build a world. If it is not, use informal methods of proof to derive a contradiction from the set.

1. *Every cube is to the left of every tetrahedron.*
2. *There are no dodecahedra.*
3. *There are exactly four cubes.*
4. *There are exactly four tetrahedra.*
5. *No tetrahedron is large.*
6. *Nothing is larger than anything to its right.*
7. *One thing is to the left of another just in case the latter is behind the former.*

14.8 ➶|✎*** (Numbers of variables) Tarski's World only allows you to use six variables. Let's explore what kind of limitation this imposes on our language.

1. Translate the sentence *There are at least two objects*, using only the predicate =. How many variables do you need?
2. Translate *There are at least three objects.* How many variables do you need?

3. It is impossible express the sentence *There are at least seven objects* using only = and the six variables available in Tarski's World, no matter how many quantifiers you use. Try to prove this. [Warning: This is true, but it is very challenging to prove. Contrast this problem with the one below.] Submit your two sentences and turn in your proof.

14.9 ✎★ (Reusing variables) In spite of the above exercise, there are in fact sentences we can express using just the six available variables that can only be true in worlds with at least seven objects. For example, in Robinson's Sentences, we give such a sentence, one that only uses the variables x and y.

1. Open this file. Build a world where there are six small cubes arranged on the front row and test the sentence's truth. Now add one more small cube to the front row, and test the sentence's truth again. Then play the game committed (incorrectly) to false. Can you see the pattern in Tarski's World's choice of objects? When it needs to pick an object for the variable x, it picks the leftmost object to the right of all the previous choices. Then, when it needs to pick an object for the variable y, it picks the last object chosen. Can you now see how the reused variables are working?
2. Now delete one of the cubes, and play the game committed (incorrectly) to true. Do you see why you can't win?
3. Now write a sentence that says there are at least four objects, one in front of the next. Use only variables x and y. Build some worlds to check whether your sentence is true under the right conditions. Submit your sentence file.

Section 14.2

Proving numerical claims

Since numerical claims can be expressed in FOL, we can use the methods of proof developed in previous chapters to prove numerical claims. However, as you may have noticed in doing the exercises, numerical claims are not always terribly perspicuous when expressed in FOL notation. Indeed, expressing a numerical claim in FOL and then trying to prove the result is a recipe for disaster. It is all too easy to lose one's grasp on what needs to be proved.

Suppose, for example, that you are told there are exactly two logic classrooms and that each classroom contains exactly three computers. Suppose you also know that every computer is in some logic classroom. From these assumptions it is of course quite easy to prove that there are exactly six computers. How would the proof go?

Proof: To prove there are exactly six computers it suffices to prove that there are at least six, and at most six. To prove that there are

Exercises

Use Fitch to give formal proofs of the following arguments. You may use **Taut Con** *where it is convenient. We urge you to work backwards, especially with the last problem, whose proof is simple in conception but complex in execution.*

14.10

$\exists x\,(\mathsf{Cube}(x) \land \forall y\,(\mathsf{Cube}(y) \to y = x))$

$\exists x\,\forall y\,(\mathsf{Cube}(y) \leftrightarrow y = x)$

14.11

$\exists x\,\forall y\,(\mathsf{Cube}(y) \leftrightarrow y = x)$

$\exists x\,(\mathsf{Cube}(x) \land \forall y\,(\mathsf{Cube}(y) \to y = x))$

14.12

$\exists x\,\exists y\,(\mathsf{Cube}(x) \land \mathsf{Cube}(y) \land x \neq y)$

$\forall x\,\forall y\,\forall z\,((\mathsf{Cube}(x) \land \mathsf{Cube}(y) \land \mathsf{Cube}(z)) \to (x = y \lor x = z \lor y = z))$

$\exists x\,\exists y\,(\mathsf{Cube}(x) \land \mathsf{Cube}(y) \land x \neq y \land \forall z\,(\mathsf{Cube}(z) \to (z = x \lor z = y)))$

14.13 ★

$\exists x\,\exists y\,(\mathsf{Cube}(x) \land \mathsf{Cube}(y) \land x \neq y \land \forall z\,(\mathsf{Cube}(z) \to (z = x \lor z = y)))$

$\exists x\,\exists y\,(\mathsf{Cube}(x) \land \mathsf{Cube}(y) \land x \neq y) \land \forall x\,\forall y\,\forall z\,((\mathsf{Cube}(x) \land \mathsf{Cube}(y) \land \mathsf{Cube}(z)) \to (x = y \lor x = z \lor y = z))$

The next two exercises contain arguments with similar premises and the same conclusion. If the argument is valid, turn in an informal proof. If it is not, submit a world in which the premises are true but the conclusion is false.

14.14 ✎

There are exactly four cubes.
Any column that contains a cube contains a tetrahedron, and vice versa.
No tetrahedron is in back of any other tetrahedron.

There are exactly four tetrahedra.

14.15 ✎

There are exactly four cubes.
Any column that contains a cube contains a tetrahedron, and vice versa.
No column contains two objects of the same shape.

There are exactly four tetrahedra.

The following exercises state some logical truths or valid arguments involving numerical quantifiers. Give informal proofs of each. Contemplate what it would be like to give a formal proof (for specific values of n and m) and be thankful we didn't ask you to give one!

14.16 ✎

$$\vdash \exists^{\leq 0} x\, S(x) \leftrightarrow \forall x\, \neg S(x)$$

[The only hard part about this is figuring out what $\exists^{\leq 0} x\, S(x)$ abbreviates.]

14.17 ✎

$$\vdash \neg\exists^{\geq n+1} x\, S(x) \leftrightarrow \exists^{\leq n} x\, S(x)$$

14.18 ✎

$\exists^{\leq n} x\, A(x)$
$\exists^{\leq m} x\, B(x)$
$\vdash \exists^{\leq n+m} x\, (A(x) \lor B(x))$

14.19 ✎

$\exists^{\geq n} x\, A(x)$
$\exists^{\geq m} x\, B(x)$
$\neg\exists x\, (A(x) \land B(x))$
$\vdash \exists^{\geq n+m} x\, (A(x) \lor B(x))$

14.20 ✎⋆

$\forall x\, [A(x) \to \exists! y\, R(x, y)]$
$\exists^{\leq n} x\, A(x)$
$\vdash \exists^{\leq n} y\, \exists x\, [A(x) \land R(x, y)]$

14.21 ✎⋆ We have seen that $\exists x\, \exists y\, R(x, y)$ is logically equivalent to $\exists y\, \exists x\, R(x, y)$, and similarly for $\forall$. What happens if we replace both of these quantifiers by some numerical quantifier? In particular, is the following argument valid?

$\exists! x\, \exists! y\, R(x, y)$
$\vdash \exists! y\, \exists! x\, R(x, y)$

If so, give an informal proof. If not, describe a counterexample.

The following exercises contain true statements about the domain of natural numbers 0, 1, ... Give informal proofs of these statements.

14.22 ✎ $\exists! x\, [x^2 - 2x + 1 = 0]$

14.23 ✎ $\exists^{!2} y\, [y + y = y \times y]$

14.24 ✎ $\exists^{!2} x\, [x^2 - 4x + 3 = 0]$

14.25 ✎ $\exists! x\, [(x^2 - 5x + 6 = 0) \land (x > 2)]$

Exercises

14.26 (The Russellian analysis of definite descriptions)

1. Open Russell's Sentences. Sentence 1 is the second of the two ways we saw in the **You try it** section on page 377 for saying that there is a single cube. Compare sentence 1 with sentence 2. Sentence 2 is the Russellian analysis of our sentence *The cube is small.* Construct a world in which sentence 2 is true.
2. Construct a world in which sentences 2-7 are all true. (Sentence 7 contains the Russellian analysis of *The small dodecahedron is to the left of the medium dodecahedron.*)

Submit your world.

14.27 (The Strawsonian analysis of definite descriptions) Using Tarski's World, open a sentence file and write the Russellian analysis of the following two sentences:

1. ***b*** *is left of the cube.*
2. ***b*** *is not left of the cube.*

Build a world containing a dodec named b and one other block in which neither of your translations is true. To do so, you will need to violate what Strawson would call the common presupposition of these two sentences. Submit both the sentence and world files.

14.28 (The Russellian analysis of *both* and *neither*) Open Russell's World. Notice that the following sentences are all true:

1. *Both cubes are medium.*
2. *Neither dodec is small.*
3. *Both cubes are in front of the tetrahedron.*
4. *Both cubes are left of both dodecahedra.*
5. *Neither cube is in back of either dodecahedron.*

Start a new sentence file and write the Russellian analysis of these five sentences. Since Tarski's World doesn't let you use the notation $\exists^{!2}$, you may find it easier to write the sentences on paper first, using this abbreviation, and then translate them into proper FOL. Check that your translations are true in Russell's World. Then make some changes to the sizes and positions of the blocks and again check that your translations have the same truth values as the English sentences.

14.29 ✎★ Discuss the meaning of the determiner *Max's*. Notice that you can say *Max's pet is happy,* but also *Max's pets are happy.* Give a Russellian and a Strawsonian analysis of this determiner. Which do you think is better?

Remember

Given any English determiner Q, we can add a corresponding quantifier Q to fol. In this extended language, the sentence Q x (A, B) is true in a world just in case Q of the objects that satisfy A(x) also satisfy B(x).

Exercises

14.30 ✎ Some of the following English determiners are reducible, some are not. If they are reducible, explain how the general form can be reduced to the special form. If they do not seem to be reducible, simply say so.

1. *At least three*
2. *Both*
3. *Finitely many*
4. *At least a third*
5. *All but one*

14.31 ✎ Open Cooper's World. Suppose we have expanded fol by adding the following expressions:

$\forall^{b1}$, meaning all but one,
Few, interpreted as meaning at most 10%, and
Most, interpreted as meaning more than half.

Translate the following sentences into this extended language. Then say which are true in Cooper's World. (You will have to use paper to write out your translations, since Tarski's World does not understand these quantifiers. If the sentence is ambiguous—for example, sentence 5—give both translations and say whether each is true.)

1. *Few cubes are small.*
2. *Few cubes are large.*
3. *All but one cube is not large.*
4. *Few blocks are in the same column as **b**.*
5. *Most things are adjacent to some cube.*
6. *A cube is adjacent to most tetrahedra.*
7. *Nothing is adjacent to most things.*
8. *Something is adjacent to something, but only to a few things.*
9. *All but one tetrahedron is adjacent to a cube.*

14.32 ✎ Once again open Cooper's World. This time translate the following sentences into English and say which are true in Cooper's World. Make sure your English translations are clear and unambiguous.

1. Most y (Tet(y), Small(y))
2. Most z (Cube(z), LeftOf(z, b))
3. Most y Cube(y)
4. Most x (Tet(x), ∃y Adjoins(x, y))
5. ∃y Most x (Tet(x), Adjoins(x, y))
6. Most x (Cube(x), ∃y Adjoins(x, y))
7. ∃y Most x (Cube(x), Adjoins(x, y))
8. Most y (y ≠ b)
9. ∀x (Most y (y ≠ x))
10. Most x (Cube(x), Most y (Tet(y), FrontOf(x, y))

Section 14.5

The logic of generalized quantification

In this section we look briefly at some of the logical properties of determiners. Since different determiners typically have different meanings, we expect them to have different logical properties. In particular, we expect the logical truths and valid arguments involving determiners to be highly sensitive to the particular determiners involved. Some of the logical properties of determiners fall into nice clusters, though, and this allows us to classify determiners in logically significant ways.

We will assume that Q is some determiner of English and that we have introduced a formal counterpart Q into FOL in the manner described at the end of the last section.

Conservativity

As it happens, there is one logical property that holds of virtually all single-word determiners in every natural language. Namely, for any predicates A and B, the following are logically equivalent:

$$\mathsf{Q\,x\,(A(x), B(x))} \;\Leftrightarrow\; \mathsf{Q\,x\,(A(x), (A(x) \land B(x)))}$$

conservativity property

This is called the *conservativity property* of determiners. Here are two instances of the ⇐ half of conservativity, followed by two instances of the ⇒ half:

*If **no** doctor is a doctor and a lawyer, then **no** doctor is a lawyer.*
*If **exactly three** cubes are small cubes, then **exactly three** cubes*

Exercises

For each of the following arguments, decide whether it is valid. If it is, explain why. This explanation could consist in referring to one of the determiner properties mentioned in this section or it could consist in an informal proof. If the argument is not valid, carefully describe a counterexample.

14.33 ✎

Few cubes are large.

Few cubes are large cubes.

14.34 ✎

Few cubes are large.

Few large things are cubes.

14.35 ✎

Many cubes are large.

Many cubes are not small.

14.36 ✎

Few cubes are large.

Few cubes are not small.

14.37 ✎

Few cubes are not small.

Few cubes are large.

14.38 ✎

Most cubes are left of *b*.

Most small cubes are left of *b*.

14.39 ✎

At most three cubes are left of *b*.

At most three small cubes are left of *b*.

14.40 ✎

Most cubes are not small.

Most cubes are large.

14.41 ✎

$\exists x\,[\mathsf{Dodec}(x) \land \mathsf{Most}\,y\,(\mathsf{Dodec}(y), y = x))]$

$\exists! x\,\mathsf{Dodec}(x)$

14.42 ✎

At least three small cubes are left of *b*.

At least three cubes are left of *b*.

14.43 ✎

Most small cubes are left of *b*.

Most cubes are left of *b*.

14.44 ✎

Most tetrahedra are left of *b*.
a is a tetrahedron in the same column as *b*.
a is not right of anything in the same row as *b*.

Most tetrahedra are not in the same row as *b*.

14.45 ✎

Only cubes are large.

Only cubes are large cubes.

14.46 ✎

Only tetrahedra are large tetrahedra.

Only tetrahedra are large.

14.47 ✎

Most of the students brought a snack to class.
Most of the students were late to class.

Most of the students were late to class and brought a snack.

14.48 ✎

Most of the students brought a snack to class.
Most of the students were late to class.

At least one student was late to class and brought a snack.

14.49 ✎⋆

Most former British colonies are democracies.
All English speaking countries were formerly British colonies.

Most English speaking countries are democracies.

14.50 ✎⋆

Many are called.
Few are chosen.

Most are rejected.

14.51 ✎⋆ In one of our example arguments, we noted that *Several A B* implies *Some A B*. In general, a determiner Q is said to have *existential import* if Q A B logically implies *Some A B*. Classify each of the determiners listed in Table 14.2 as to whether it has existential import. For those that don't, give informal counterexamples. Discuss any cases that seem problematic.

14.52 ✎ Consider a hypothetical English determiner "allbut." For example, we might say *Allbut cubes are small* to mean that all the blocks except the cubes are small. Give an example to show that "allbut" is not conservative. Is it monotone increasing or decreasing? Persistent or anti-persistent? Illustrate with arguments expressed in English augmented with "allbut."

14.53 ✎⋆ (Only) Whether or not *only* is a determiner, it could still be added to FOL, allowing expressions of the form Only x (A, B), which would be true if and only if only A's are B's.

1. While *Only* is not conservative, it does satisfy a very similar property. What is it?
2. Discuss monotonicity and persistence for *Only*. Illustrate your discussion with arguments expressed in English.

14.54 ✎⋆ (Adverbs of temporal quantification) It is interesting to extend the above discussion of quantification from determiners to so-called adverbs of temporal quantification, like *always*, *often*, *usually*, *seldom*, *sometimes*, and *never*. To get a hint how this might go, let's explore the ambiguities in the English sentence *Max usually feeds Carl at 2:00 p.m.*

Earlier, we treated expressions like 2:00 as names of times on a particular day. To interpret this sentence in a reasonable way, however, we need to treat such expressions as predicates of times. So we need to add to our language a predicate 2pm(t) that holds of those times t (in the domain of discourse) that occur at 2 p.m., no matter on what day they occur. Let us suppose that *Usually* means most times. Thus,

$$\mathsf{Usually}\,\mathsf{t}\,(\mathsf{A(t)}, \mathsf{B(t)})$$

means that most times satisfying $\mathsf{A(t)}$ also satisfy $\mathsf{B(t)}$.

1. One interpretation of *Max usually feeds Carl at 2:00 p.m.* is expressed by

   $$\mathsf{Usually}\,\mathsf{t}\,(\mathsf{2pm(t)}, \mathsf{Feeds(max, carl, t)})$$

   Express this claim using an unambiguous English sentence.
2. A different interpretation of the sentence is expressed by

   $$\mathsf{Usually}\,\mathsf{t}\,(\mathsf{Feeds(max, carl, t)}, \mathsf{2pm(t)})$$

   Express this claim using an unambiguous English sentence. Then elucidate the difference between this claim and the first by describing situations in which each is true while the other isn't.
3. Are the same ambiguities present in the sentence *Claire seldom feeds Folly at 2:00 p.m.*? How about with the other adverbs listed above?
4. Can you think of yet a third interpretation of *Max usually feeds Carl at 2:00 p.m.*, one that is not captured by either of these translations? If so, try to express it in our language or some expansion of it.

Section 14.6

Other expressive limitations of first-order logic

The study of generalized quantification is a response to one expressive limitation of fol, and so to its inability to illuminate the full logic inherent in natural languages like English. The determiners studied in the preceding sections are actually just some of the ways of expressing quantification that we find in natural languages. Consider, for example, the sentences

> ***More*** *cubes* ***than*** *tetrahedra are on the same row as e.*
> ***Twice as many*** *cubes* ***as*** *tetrahedra are in the same column as f.*
> ***Not as many*** *tetrahedra* ***as*** *dodecahedra are large.*

three place quantification

The expressions in bold take two common noun expressions and a verb expression to make a sentence. The techniques used to study generalized quantification in earlier sections can be extended to study these determiners, but we have to think of them as expressing *three place* relations on sets, not just two place relations. Thus, if we added these determiners to the language, they would have the general form $\mathsf{Q}\,\mathsf{x}\,(\mathsf{A(x)}, \mathsf{B(x)}, \mathsf{C(x)})$.

A related difference in expressive power between fol and English comes in the ability of English to use both singular and plural noun phrases. There is a difference between saying *The boys argued with the teacher* and saying *Every boy argued with the teacher.* The first describes a single argument between a teacher and a group of boys, while the second may describe a sequence of distinct arguments. Fol does not allow us to capture this difference.

plurals

Quantification is just the tip of an iceberg, however. There are many expressions of natural languages that go beyond first-order logic in various ways. Some of these we have already discussed at various points, both with examples and exercises. As one example, we saw that there are many uses of the natural language conditional *if... then...* that are not truth functional, and so not captured by the truth-functional connective $\rightarrow$.

Another dimension in which fol is limited, in contrast to English, comes in the latter's flexible use of tense. Fol assumes a timeless domain of unchanging relationships, whereas in English, we can exploit our location in time and space to say things about the present, the past, and locations around us. For example, in fol we cannot easily say that it is hot here today but it was cool yesterday. To say something similar in fol, we need to allow quantifiers over times and locations, and add corresponding argument positions to our atomic predicates.

tense

Similarly, languages like English have a rich modal structure, allowing us not only to say how things are, but how they must be, how they might be, how they can't be, how they should be, how they would be if we had our way, and so forth. So, for example, we can say *All the king's horses couldn't put Humpty Dumpty together again.* Or *Humpty shouldn't have climbed on the wall.* Or *Humpty might be dead.* Such statements lie outside the realm of fol.

modality

All of these expressions have their own logic, and we can explore and try to understand just which claims involving these expressions follow logically from others. Building on the great success of fol, logicians have studied (and are continuing to study) extensions of fol in which these and similar expressive deficiencies are addressed. But as of now there is no single such extension of fol that has gained anything like its currency.

Exercises

14.55 ✎ Try to translate the nursery rhyme about Humpty Dumpty into fol. Point out the various linguistic mechanisms that go beyond fol. Discuss this in class.

14.56 ✎ Consider the following two claims. Does either follow logically from the other? Are they logically equivalent? Explain your answers.

1. I can eat every apple in the bowl.
2. I can eat any apple in the bowl.

14.57 ✎⋆ Recall the first-order language introduced in Table 1.2, page 30. Some of the following can be given first-order translations using that language, some cannot. Translate those that can be. For the others, explain why they cannot be faithfully translated, and discuss whether they could be translated with additional names, predicates, function symbols, and quantifiers, or if the shortcoming in the language is more serious.

1. *Claire gave Max at least two pets at 2:00 pm.*
2. *Claire gave Max at most two pets at 2:00 pm.*
3. *Claire gave Max several pets at 2:00 pm.*
4. *Claire was a student before Max was.*
5. *The pet Max gave Claire at 2:00 pm was hungry.*
6. *Most pets were hungry at noon.*
7. *All but two pets were hungry at noon.*
8. *There is at least one student who made Max angry every time he (or she) gave Max a pet.*
9. *Max was angry whenever a particular student gave him a pet.*
10. *If someone gave Max a pet, it must have been Claire.*
11. *No pet fed by Max between 2:00 and 2:05 belonged to Claire.*
12. *If Claire fed one of Max's pets before 2:00 pm, then Max was angry at 2:00 pm.*
13. *Folly's owner was a student.*
14. *Before 3:00, no one gave anyone a pet unless it was hungry.*
15. *No one should give anyone a pet unless it is hungry.*
16. *A pet that is not hungry always belongs to someone or other.*
17. *A pet that is not hungry must belong to someone or other.*
18. *Max was angry at 2:00 pm because Claire had fed one of his pets.*
19. *When Max gave Folly to Claire, Folly was hungry, but Folly was not hungry five minutes later.*
20. *No student could possibly be a pet.*

14.58 ✎⋆⋆⋆ Here is a famous puzzle. There was a Roman who went by two names, "Cicero" and "Tully." Discuss the validity or invalidity of the following argument.

Bill claims Cicero was a great orator.
Cicero is Tully.

Bill claims Tully was a great orator.

What is at stake here is nothing more or less than the principle that if $(\ldots\ a \ldots)$ is true, and $a = b$, then $(\ldots\ b \ldots)$ is true. [Hint: Does the argument sound more reasonable if we replace "claims" by "claims that"? By the way, the puzzle is usually stated with "believes" rather than "claims."]

The following more difficult exercises are not specifically relevant to this section, but to the general topic of truth of quantified sentences. They can be considered as research projects in certain types of classes.

14.59 ✎⋆⋆ (Persistence through expansion) As we saw in Exercise 11.5, page 301, some sentences simply can't be made false by adding objects of various sorts to the world. Once they are true, they stay true. For example, the sentence *There is at least one cube and one tetrahedron*, if true, cannot be made false by adding objects to the world. This exercise delves into the analysis of this phenomenon in a bit more depth.

Let's say that a sentence A is *persistent through expansion* if, whenever it is true, it remains true no matter how many objects are added to the world. (In logic books, this is usually called just persistence, or persistence under extensions.) Notice that this is a semantic notion. That is, it's defined in terms of truth in worlds. But there is a corresponding syntactic notion. Call a sentence *existential* if it is logically equivalent to a prenex sentence containing only existential quantifiers.

- Show that $\mathsf{Cube(a)} \rightarrow \exists \mathsf{x}\, \mathsf{FrontOf(x, a)}$ is an existential sentence.
- Is $\exists \mathsf{x}\, \mathsf{FrontOf(x, a)} \rightarrow \mathsf{Cube(a)}$ an existential sentence?
- Show that every existential sentence is persistent through expansion. [Hint: You will have to prove something slightly stronger, by induction on wffs. If you are not familiar with induction on wffs, just try to understand why this is the case. If you are familiar with induction, try to give a rigorous proof.] Conclude that every sentence equivalent to an existential sentence is persistent through expansion.

It is a theorem, due to Tarski and Łoś (a Polish logician whose name is pronounced more like "wash" than like "loss"), that any sentence that is persistent through expansion is existential. Since this is the converse of what you were asked to prove, we can conclude that a sentence is persistent through expansion if and only if it is existential. This is a classic example of a theorem that gives a syntactic characterization of some semantic notion. For a proof of the theorem, see any textbook in model theory.

14.60 ➶ (Invariance under motion, part 1) The real world does not hold still, the way the world of mathematical objects does. Things move around. The truth values of some sentences change with such motion, while the truth values of other sentences don't. Open Ockham's World and Ockham's Sentences. Verify that all the sentences are true in the given world. Make as many of Ockham's Sentences false as you can by just moving objects around. Don't add or remove any objects from the world, or change their size or shape. You should be able to make false (in a single world) all of the sentences containing any spatial predicates, that is, containing LeftOf, RightOf, FrontOf, BackOf, or Between. (However, this is a quirk of this list of sentences, as we will see in the next exercise.) Save the world as World 14.60.

14.61 ✎★★ (Invariance under motion, part 2) Call a sentence *invariant under motion* if, for every world, the truth value of the sentence (whether true *or* false) does not vary as objects move around in that world.

1. Prove that if a sentence does not contain any spatial predicates, then it is invariant under motion.
2. Give an example of a sentence containing a spatial predicate that is nonetheless invariant under motion.
3. Give another such example. But this time, make sure your sentence is not first-order equivalent to any sentence that doesn't contain spatial predicates.

14.62 ➶ (Persistence under growth, part 1) In the real world, things not only move around, they also grow larger. (Some things also shrink, but ignore that for now.) Starting with Ockham's World, make the following sentences true by allowing some of the objects to grow:

1. $\forall x \neg \mathsf{Small}(x)$
2. $\exists x \exists y (\mathsf{Cube}(x) \land \mathsf{Dodec}(y) \land \mathsf{Larger}(y, x))$
3. $\forall y (\mathsf{Cube}(y) \rightarrow \forall v (v \neq y \rightarrow \mathsf{Larger}(v, y)))$
4. $\neg \exists x \exists y (\neg \mathsf{Large}(x) \land \neg \mathsf{Large}(y) \land x \neq y)$

How many of Ockham's Sentences are false in this world? Save your world as World 14.62.

14.63 ✎★ (Persistence under growth, part 2) Say that a sentence S is *persistent under growth* if, for every world in which S is true, S remains true if some or all of the objects in that world get larger. Thus, Large(a) and ¬Small(a) are persistent under growth, but Smaller(a, b) isn't. Give a syntactic definition of as large a set of sentences as you can for which every sentence in the set is persistent under growth. Can you prove that all of these sentences are persistent under growth?