Remember

Given an inductive definition of a set, an inductive proof requires

- a *basis step,* which shows that the property holds of the basic elements, and
- an *inductive step,* which shows that *if* the property holds of some elements, then it holds of any elements generated from them by the inductive clauses.

The assumption that begins the inductive step is called the *inductive hypothesis.*

Exercises

16.1 ✎ In the state of Euphoria, the following two principles hold:

1. If it is sunny on one day, it is sunny the next day.
2. It is sunny today.

Prove that it is going to be sunny from now on.

16.2 ✎ Raymond Smullyan, a famous logician/magician, gives the following good advice: (1) always speak the truth, and (2) each day, say "I will repeat this sentence tomorrow." Prove that anyone who did these two things would live forever. Then explain why it won't work.

16.3 ✎ Give at least two distinct derivations which show that the following is an ambig-wff: $A_1 \rightarrow A_2 \leftrightarrow \neg A_2$.

16.4 ✎ Prove by induction that no ambig-wff begins with a binary connective, ends with a negation sign, or has a negation sign immediately preceding a binary connective. Conclude that the string $A_1 \neg \rightarrow A_2$ is not an ambig-wff.

16.5 ✎ Prove that no ambig-wff ever has two binary connectives next to one another. Conclude that $A_1 \rightarrow \vee A_2$ is not an ambig-wff.

16.6 ✎ Modify the inductive definition of ambig-wff as follows, to define the set of semi-wffs:

1. Each propositional letter is a semi-wff.
2. If p is any semi-wff, so is the string $\neg p)$.
3. If p and q are semi-wffs, so are $(p \wedge q), (p \vee q), (p \rightarrow q), (p \leftrightarrow q)$.
4. Nothing is a semi-wff except in virtue of repeated applications of (1), (2), and (3).

Prove by induction that every semi-wff has the following property: the number of right parentheses is equal to the number of left parentheses plus the number of negation signs.

16.7 ✎ In the text, we proved that every pal is a palindrome, a string of letters that reads the same back to front and front to back. Is the converse true, that is, is every palindrome a pal? If so, prove it. If not, fix up the definition so that it becomes true.

16.8 ✎ (Existential wffs) In this problem we return to a topic raised in Problem 14.59. In that problem we defined an existential sentence as one whose prenex form contains only existential quantifiers. A more satisfactory definition can be given by means of the following inductive definition. The existential wffs are defined inductively by the following clauses:

1. Every atomic or negated atomic wff is existential.
2. If $\mathsf{P}_1, \ldots, \mathsf{P}_n$ are existential, so are $(\mathsf{P}_1 \lor \ldots \lor \mathsf{P}_n)$ and $(\mathsf{P}_1 \land \ldots \land \mathsf{P}_n)$.
3. If P is an existential wff, so is $\exists \nu P$, for any variable ν.
4. Nothing is an existential wff except in virtue of (1)–(3).

Prove the following facts by induction:

- If P is an existential wff, then it is logically equivalent to a prenex wff with no universal quantifiers.
- Suppose P is an existential sentence of the blocks language. Prove that if P is true in some world, then it will remain true if new objects are added to the world. [You will need to prove something a bit stronger to keep the induction going.]

Is our new definition equivalent to our old one? If not, how could it be modified to make it equivalent?

16.9 ✎ Give a definition of universal wff, just like that of existential wff in the previous problem, but with universal quantifiers instead of existential. State and prove results analogous to the results you proved there. Then show that every universal wff is logically equivalent to the negation of an existential wff.

16.10 ✎★★ Define the class of *wellfounded sets* by means of the following inductive definition:

1. If C is any set of objects, each of which is either not a set or is itself a wellfounded set, then C is a wellfounded set.
2. Nothing is a wellfounded set except as justified by (1).

This exercise explores the relationship between the wellfounded sets and the cumulative conception set discussed in the preceding chapter.

1. Which of the following sets are wellfounded?

$$\emptyset, \{\emptyset\}, \{\text{Washington Monument}\}, \{\{\{\ldots\}\}\}$$

2. Assume that a is wellfounded. Show that $\wp a$ is wellfounded.
3. Assume that a and b are wellfounded. Is the ordered pair $\langle a, b \rangle$ (as defined in the preceding chapter) wellfounded?
4. Assume that $a = \{1, b\}$ and $b = \{2, a\}$. Are a and b wellfounded?
5. Show that the Axiom of Regularity implies that every set is wellfounded.
6. When using set theory, one often wants to be able to prove statements of the form:

$$\forall x\, [\mathsf{Set}(x) \rightarrow \mathsf{Q}(x)]$$

One of the advantages of the cumulative conception of set discussed in the preceding chapter is that it allows one to prove such statements "by induction on sets." How?
7. Use mathematical induction to show that there is no infinite sequence of wellfounded sets $a_1, a_2, a_3, \ldots$ such that $a_{n+1} \in a_n$ for each natural number n.

Section 16.2

Inductive definitions in set theory

The way we have been stating inductive definitions seems reasonably rigorous. Still, you might wonder about the status of clauses like

4. Nothing is an ambig-wff unless it can be generated by repeated applications of (1), (2), and (3).

This clause is quite different in character from the others, since it mentions not just the objects we are defining, but the other clauses of the definition itself. You might also wonder just what is getting packed into the phrase "repeated applications."

One way to see that there is something different about clause (4) is to note that the other clauses are obviously expressible using first-order formulas. For example, if concat is a symbol for the concatenation function (that is, the function that takes two expressions and places the first immediately to the left of the second), then one could express (2) as

$$\forall p\ [\mathsf{ambig\text{-}wff}(p) \rightarrow \mathsf{ambig\text{-}wff}(\mathsf{concat}(\neg, p))]$$

In contrast, clause (4) is not the sort of thing that can be expressed in fol.

making the final clause more precise

However, it turns out that if we work within set theory, then we can express inductive definitions with first-order sentences. Here, for example, is a definition of the set of ambig-wffs that uses sets. It turns out that this definition can be transcribed into the language of set theory in a straightforward way. The English version of the definition is as follows:

you take the intersection of a bunch of sets, the result is always a subset of all of the original sets.

The situation is illustrated in Figure 16.1. There are lots of sets that satisfy clauses (1)–(3) of our definition, most of which contain many elements that are not ambig-wffs. For example, the set of all finite strings of propositional letters and connectives satisfies (1)–(3), but it contains strings like $A_1 \neg \rightarrow A_2$ that aren't ambig-wffs. Our set theoretic definition takes the set S of ambig-wffs to be the *smallest*, that is, the *intersection* of all these sets.

justifying induction

Notice that we can now explain exactly why proof by induction is a valid form of reasoning. When we give an inductive proof, say that all ambig-wffs have property Q, what we are really doing is showing that the set $\{x \mid Q(x)\}$ satisfies clauses (1)–(3). We show that the basic elements all have property Q and that if you apply the generation rules to things that have Q, you will get other things that have Q. But if Q satisfies clauses (1)–(3), and S is the intersection of all the sets that satisfy these clauses, then $S \subseteq Q$. Which is to say: all ambig-wffs have property Q.

Exercises

16.11 ✎ Prove Lemma 3.

16.12 ✎ Give an inductive definition of the set of wffs of propositional logic, similar to the above definition, but putting in the parentheses in clause (3). That is, the set of wffs should be defined as the smallest set satisfying various clauses. Be sure to verify that there is such a smallest set.

16.13 ✎ Based on your answer to Exercise 16.12, prove that every wff has the same number of left parentheses as binary connectives.

Section 16.3

Induction on the natural numbers

Many students come away from the study of induction in math classes with the feeling that it has something special to do with the natural numbers. By now, it should be obvious that this method of proof is far more general than that. We can prove things about many different kinds of sets using induction. In fact, whenever a set is defined inductively, we can prove general claims might check to see that $Q(1)$ holds. You might even go so far as to check $Q(2)$, although it's not necessary.)

Induction: To prove the inductive step, we assume that we have a natural number k for which $Q(k)$ holds, and show that $Q(k+1)$ holds. That is, our inductive hypothesis is that the sum of the first k natural numbers is $k(k-1)/2$. We must show that the sum of the first $k+1$ natural numbers is $k(k+1)/2$. How do we conclude this? We simply note that the sum of the first $k+1$ natural numbers is k greater than the sum of the first k natural numbers (since the first natural number is zero, the second is one, and so on). We already know by the inductive hypothesis that this latter sum is simply $k(k-1)/2$. Thus the sum of the first $k+1$ numbers is

$$\frac{k(k-1)}{2} + k$$

Getting a common denominator gives us

$$\frac{k(k-1)}{2} + \frac{2k}{2}$$

which we factor to get

$$\frac{k(k+1)}{2}$$

the desired result.

Exercises

16.14 ✎ Prove by induction that for all natural numbers n, $n \leq 2n$.

16.15 ✎ Prove by induction that for all natural numbers n, $0 + 1 + \ldots + n \leq n^2$. Your proof should not presuppose Proposition 4, which we proved in the text, though it will closely follow the structure of that proof.

16.16 ✎ Prove by induction that for all n,

$$1 + 3 + 5 + \ldots + (2n+1) = (n+1)^2$$

16.17 ✎⋆ Prove that for all natural numbers $n \geq 2$,

$$(1 - \frac{1}{2})(1 - \frac{1}{3}) \ldots (1 - \frac{1}{n}) = \frac{1}{n}$$

16.18 ✎★ Notice that $1^3+2^3+3^3=36=6^2$ and that $1^3+2^3+3^3+4^3+5^3=225=15^2$. Prove that the sum of the first n perfect cubes is a square. [Hint: This is an instance of the inventor's paradox. You will have to prove something stronger than this.]

16.19 ✎ (after Pólya) Examine the following incorrect proof that all logicians have the same shoe size. Identify and describe clearly why the proof is incorrect.

Basis: Every set containing just one logician, contains logicians all of whom have the same shoe size.

Induction: Our induction hypothesis is that any set of n logicians contains logicians with the same shoe size. Now look at any set of $n+1$ logicians. We must show that this set contains logicians with the same shoe size. Number the logicians: $1, 2, 3, \ldots, n, n+1$, and look at the sets $\{1, 2, 3, \ldots, n\}$ and $\{2, 3, 4, \ldots, n+1\}$. Each of these sets contain only n logicians, therefore within each set all of the logicians have the same shoe size. But the two sets overlap, so there must be only one shoe size among all $n+1$ logicians.

Section 16.4

Axiomatizing the natural numbers

In giving examples of informal proofs in this book, we have had numerous occasions to use the natural numbers as examples. In proving things about the natural numbers, we have made recourse to any fact about the natural numbers that was obviously true. If we wanted to formalize these proofs, we would have to be much more precise about what we took to be the "obvious" facts about the natural numbers.

Peano Arithmetic (PA)

Over the years, a consensus has arisen that the obviously true claims about the natural numbers can be formalized in what has come to be known as Peano Arithmetic, or PA for short, named after the Italian mathematician Giuseppe Peano. This is a certain first-order theory whose main axiom states a form of induction for natural numbers.

successor function

PA is formulated in a first-order language that has the constant 0 together with the unary successor function s and the identity predicate. It also contains the binary function symbols $+$ and $\times$, but we think of 0 and s as primitive, while the meanings of $+$ and $\times$ will be completely defined by axioms of PA. Intuitively, the successor function applied to any number n gives us the next greatest number (what we would normally write as $n+1$). So $s(0)$ denotes 1, $s(s(0))$ denotes 2, and so forth.

Here are the first six axioms of Peano Arithmetic. We will get to the important induction axiom in due course.

Exercises

Give informal proofs, similar in style to the one in the text, that the following statements are consequences of PA. *Explicitly identify any predicates to which you apply induction. When proving the later theorems, you may assume the results of the earlier problems.*

16.20 ✎ $\forall x\ (0 + x = x)$

16.21 ✎ $\forall x\ (s(0) \times x = x)$

16.22 ✎ $\forall x\ (0 \times x = 0)$

16.23 ✎⋆ $\forall x\ (x \times s(0) = s(0) \times x)$

16.24 ✎ $\forall x\ \forall y\ (x + s(y) = s(x) + y)$ [Hint: Don't get confused by the two universal quantifiers. Start by assuming that x is an arbitrary number and perform induction on y. This does not require double induction.]

16.25 ✎ $\forall x\, \forall y\, \forall z\, (x + z = y + z \rightarrow x = y)$ [Hint: this is an easy induction on z.]

16.26 ✎⋆ $\forall x\, \forall y\, \forall z\, ((x + y) + z = x + (y + z))$ [Hint: This is relatively easy, but you have to perform induction on z. That is, your basis case is to show that $(x + y) + 0 = x + (y + 0)$. You should then assume $(x + y) + n = x + (y + n)$ as your inductive hypothesis and show $(x + y) + s(n) = x + (y + s(n))$.]

16.27 ✎⋆⋆ $\forall x\, \forall y\, (x + y = y + x)$ [Hint: this requires double induction.]

16.28 ✎⋆⋆ $\forall x\, \forall y\, (x \times y = y \times x)$ [Hint: To prove this, you will first need to prove the lemma $\forall x\, \forall y\, (s(x) \times y = (x \times y) + y)$. Prove this by induction on y.]

Section 16.5
Induction in Fitch

Fitch contains an inference rule for induction on the natural numbers which exactly parallels the informal technique that we have just described. In order to complete a formal proof by induction of a universally quantified formula, you must be able to cite the statement of the base case of the induction, and a subproof which represents the step case:

16.29 $\forall x\ (0 + x = x)$

16.30 $\forall x\ (s(0) \times x = x)$

16.31 $\forall x\ (0 \times x = 0)$

16.32 $\forall x\ (x \times s(0) = s(0) \times x)$

16.33 $\forall x\ \forall y\ (x + s(y) = s(x) + y)$ [Hint: This does not require double induction. Your final step will be universal generalization on x, but within the subproof leading up to that step, you will need to perform induction on y.]

16.34 $\forall x\,\forall y\,\forall z\,((x + y) + z = x + (y + z))$ [Hint: Prove this by induction on z.]

16.35 $\forall x\,\forall y\,\forall z\,(x + z = y + z \rightarrow x = y)$

16.36 $\forall x\,\forall y\,(x + y = y + x)$ [Hint: this requires double induction.]

16.37 $\forall x\ \forall y\ (s(x) \times y = (x \times y) + y)$ [Hint: This is the key lemma that you will need to prove Exercise 16.38. In order to prove it, you will need the Associativity and Commutativity of Addition (Exercises 16.34 and 16.36, respectively).]

16.38 $\forall x\,\forall y\,(x \times y = y \times x)$ [Hint: This is pretty easy, once you have Exercises 16.31 and 16.37 to use as lemmas.]

Section 16.6

Ordering the Natural Numbers

We usually think of the natural numbers distributed along a number line with 0 on the left and the numbers increasing toward the right. A number further to the left is "less than" any number to its right. We can express this idea formally using the following axiom, which defines the relation $<$.

$$\forall x\ \forall y\ (x < y \leftrightarrow \exists z\ (x + s(z) = y))$$

This says that x is less than y whenever we can obtain y by adding some non-zero number to x. We will see that the relation $<$ plays an important role in a second form of induction that we will describe in the next section. First though, lets investigate the properties of this relation.

Any relation, R, which has the following three properties is called a *total strict ordering* (TSO):

Proposition 7. Trichotomy

$$\forall x\,\forall y\,(x < y \lor x = y \lor y < x)$$

Proof: Let a be arbitrary. We will show that either

$$\forall y\,(a < y \lor a = y \lor y < a)$$

by induction on y.

Basis: We must show $(a < 0 \lor a = 0 \lor 0 < a)$. We know that every number, including a, is either 0, or the successor of some number. 0 is less than any successor, so one of the second two disjuncts must hold.

Induction: We assume that $(a < k \lor a = k \lor k < a)$ for some k and show that $(a < s(k) \lor a = s(k) \lor s(k) < a)$. Let's split into cases according to the disjuncts of the induction hypothesis.

$a < k$ In this case $a < s(k)$, and we are done.

$a = k$ In this case too, $a < s(k)$.

$k < a$ This tells us that $\exists y\, k + s(y) = a$. y must be either 0 or the successor of some number. If $y = 0$, then $k + s(0) = a$ or equivalently $s(k) = a$. If on the other hand $y = s(b)$ for some b, then $k + s(s(b)) = a$, which means that $s(k) + s(b) = a$, and so $s(k) < a$ (see Exercise 16.24).

Exercises

Use Fitch to construct formal proofs of the following theorems from the Peano Axioms plus the definition of <. In the corresponding Exercise file, you will find as premises only the specific axioms needed to prove the goal theorem.

16.39 $\forall x\, \neg x < 0$

16.40 $\forall x\, x < s(x)$

16.41 $\forall x\, \exists y\, x < y$ [Hint: Notice that the Exercise file contains the definition of < but none of the Peano Axioms! So although this follows from Exercise 16.40, you won't be able to use your proof of that that as a lemma. The proof is actually pretty simple, but requires some thought.]

16.42 $\forall x \forall y (x < y \rightarrow x < s(y))$

For the following exercises you will want to use your proofs from Section 16.5 as lemmas.

16.43 $\forall x (x = 0 \lor 0 < x)$

16.44 $\forall x \forall y (x < y \rightarrow s(x) < s(y))$

16.45 $\forall x \neg x < x$

16.46 $\forall x \forall y \forall z ((x < y \land y < z) \rightarrow x < z)$

16.47 ★★ $\forall x \forall y (x < y \lor x = y \lor y < x)$ [Hint: You'll find some earlier exercises from this section to be handy lemmas for this proof.]

16.48 Define $\leq$ so that $\forall x \forall y (x \leq y \leftrightarrow \exists z\, x + z = y)$. Give informal proofs that $\leq$ is reflexive and transitive.

16.49 Define $\geq$ so that $\forall x \forall y (x \geq y \leftrightarrow \neg y < x)$. Give informal proofs that $\geq$ is reflexive and transitive.

Section 16.7

Strong Induction

Sometimes the induction principle that we have described does not fit well with the property of natural numbers that we are trying to prove. As an example of this imagine proving that every natural number greater than one is either prime, or can be expressed as the product of primes. Mathematicians call this fact *The Fundamental Theorem of Arithmetic*. Anything that merits such a grand name deserves an investigation. Let's see what happens when we try to prove this using ordinary induction on the natural numbers. We will start with a base case of $n = 2$, since the claim does not apply to 0 or 1.

Proposition 8. For any number n greater than 1, n is either prime or can be expressed as a product of primes, i.e., $n = p_1 \times \ldots \times p_m$, where each of the p's are prime.

You might think that strong induction is poorly named, since it follows from ordinary "weak" induction. But the point of the name is not that the principle is stronger than ordinary induction. In fact, anything you can prove by one you can also prove by the other. The difference is simply that strong induction allows you to use a stronger inductive hypothesis. You get to assume that all the numbers smaller than n have the property, not just its immediate predecessor.

You should probably be able to guess the form of Fitch's strong induction rule:

Strong Induction:

	| [n] $\forall x\,(x < n \to P(x))$	
	| ⋮	Where n does not occur outside the subproof where it is introduced.
	| $P(n)$	
▷	$\forall x\, P(x)$	

Exercises

The following exercises work together to result in a formal proof of Proposition 9. You'll need to do them all to get to the final proof, but it will be worth the work.

16.50 We begin by proving a simple lemma, namely that every number is either 0 or the successor of some other number. If you completed exercise 16.43, then you already proved something similar.

$$\forall x\ (x = 0 \lor \exists y\ x = s(y))$$

16.51 You may have thought that we would use the lemma in the previous exercise in the proof of Proposition 9, but in fact we are going to use it to prove a second lemma. In the informal proof of Proposition 9 we appealed twice to the following fact:

$$\forall x\ \forall y\ (x < s(y) \to (x = y \lor x < y))$$

Prove this fact, using lemma 16.50 if necessary.

16.52 Open the file Lemma 16.52. This file contains the skeleton of the proof of the inductive step in Proposition 9. Complete the proof by selecting the correct rules, and citations for all of the steps. Submit your file as Proof 16.52.

16.53 Finally open the file Exercise 16.53. This asks you to prove Proposition 9. Formalize the informal proof of this result that we gave in the previous section, using the lemma from Exercise 16.52 where necessary. You may use **Fo Con**, but if you like a challenge you can try to complete the proof without using it.