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A Semantical Analysis of Focusing and Contraction in Intuitionistic Logic

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Abstract. Focusing is a proof-theoretic device to structure proof search in the sequent calculus: it provides a normal form to cut-free proofs in which the application of invertible and non-invertible inference rules is structured in two separate and disjoint phases. Although stemming from proof-search considerations, focusing has not been thoroughly investigated in actual theorem proving, in particular w.r.t. termination. We present a contraction-free (and hence terminating) focused multi-succedent sequent calculus for propositional intuitionistic logic, which refines the $\mathbf{G4ip}$ calculus in the tradition of Vorob'ev, Hudelmeier and Dyckhoff. We prove completeness of the calculus semantically and argue that this offers a viable alternative to other more syntactical means.

1. Introduction

Focusing [1] is a proof-theoretic device to structure proof search in the sequent calculus: it provides a more refined normal form to cut-free proofs in which the application of invertible and non-invertible inference rules is structured in two separate and disjoint phases: in the first one, called *negative* or *asynchronous*, we apply (reading the proof bottom up) all invertible inference rules in whatever order, until none is left. The second phase, called *positive* or *synchronous*, "focuses" on a formula, by selecting a not (necessarily) invertible inference rule. If after the (reverse) application of that introduction rule, a sub-formula of the focused formula appears also requiring a non-invertible inference rule, then the phase continues with that sub-formula as the new focus. This phase ends either with success or when only formulas with invertible inference rules are encountered and phase one is re-entered. Certain "structural" rules are used to recognize the phase change, namely from negative to positive and vice versa.

Focusing was first introduced by Andreoli [1] to give a logic programming interpretation to full linear logic; as such, it naturally fits other logics with strong dualities, such as classical logic. W.r.t. intuitionistic logic, the picture is more complex; while we can find in the literature calculi that we can now recognize as exhibiting some focusing behavior, namely uniform proofs [25] and **LJT** [19] for backward chaining proofs, and **LJQ'/LJQ*** [11] for forward chaining, it was not until **LJF** [21] that the situation was satisfactory settled by Liang and Miller. This is particularly striking compared to previous text-book presentation such as [31], where a search strategy for an intuitionistic (multi-succedent) sequent calculus is described by simply dividing rules in groups to be applied following some priorities and constraints. This without a proof of completeness. Focusing instead internalizes in the proof-theory a stringent search strategy, and a provably complete one, from which many additional optimizations may follow.

Given our track record [4, 13–15], we are particularly interested in *decision* procedures for intuitionistic propositional logic; from that standpoint, one of Gentzen's original structural rules is particularly worrisome: *contraction* (or duplication, seen from the bottom up), which permits the reuse of a formula in the antecedent or succedent of a sequent:

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,\mathrm{Contr}\ L \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,\mathrm{Contr}\ R$$

The reason is clear: during proof search contraction can be applied at any time making termination problematic even for a logic that is known to be decidable. There are of course techniques that can deal with this issue, such as histories or loop detection, but they tend to be expensive and more crucially non-logical. In the Fifties Kleene made a first step by introducing the **G3** sequents systems, where the structural rules where absorbed, though not eliminated. This analysis showed how in intuitionistic propositional logic contraction is only required in the $\to L$ rule[1]. This opened the road in the Sixties to the first presentation of a *terminating* contraction-free calculus, known as **G4ip**, as developed by Vorob'ev, Hudelmeier and Dyckhoff (VHD). We refer to [10, 11, 30] for more on the history of the subject. The main idea is to replace the $\to L$ rule by a series of rules depending on the shape of the subformula A of the main formula $A \to B$ of the original rule. It is by now routine to show that such a system is indeed terminating, in the sense that any bottom-up derivation of a given sequent is of finite length. With some additional effort, one can also prove that contraction is admissible in such a calculus [12].

Focusing has already had something to say on the subject of structural rules, although not so much w.r.t. linear logic, where this is handled by exponentials nor w.r.t. classical logic, where contraction is easily seen to be *admissible* and as such does not affect proof search. As far as (mono-succedent) intuitionistic logic is concerned, a remarkable corollary of the completeness of focusing is that contraction is exactly located in between the asynchronous and synchronous phases, that is in the *Focus*^{L} rule, and is restricted to negative formulas; this means that positive non-atomic formulas on the left and negative ones on the right can be treated *linearly*.

One can argue that it should be fairly immediate to focalize a system such as **G4ip**: after all, we just need to appropriately classify as positive/negative the new left implication rules. However, the design of a focused decision procedure is quite delicate, as we need to take particular care not to create an infinite loop in case of failure. This means disciplining the structural rules of focusing, which, having been so far concerned uniquely with proofs and provability, tend to be particularly unruly.

[1] A terminating variant of **G3** is presented in [14].

We also add another twist, to test the commonly-held hypothesis that every "reasonable" sequent calculus has a natural focused version: rather than **LJ**, we consider here Maehara's multiple-conclusion sequent calculus [23], where in contrast with **LK**, the rules for right implication (and universal quantification) can only be performed if there is a single formula in the succedent of the premise to which those rules are applied. As these are the same connectives that are "global" in the Kripke semantics, this historically opened up a fecund link with tableaux systems. Moreover, Maehara's **LB** (following [31]'s terminology) has more symmetries from the permutation point of view and therefore may seem a better candidate for focusing than mono-succedent **LJ**. The two crucial rules are, unsurprisingly, the rules for implication:

$$\frac{\Gamma, A \to B \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \to L \qquad\qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B, \Delta} \to R$$

Interestingly here, in stark opposition to **LJ**, the $\to L$ rule *is* invertible, while $\to R$ is not. According to the focusing diktat, implication would be classified as a *positive* connective, leaving us with the unappealing choice between an incomplete calculus, where $A \to B$ is not duplicated, or an endless asynchronous phase, where $A \to B$ is. Note further that in **LB** the restriction of contraction to negative formulas does not apply, since, as we have seen, we need to duplicate both positive (e.g. implications) and negative formulas (additive conjunction, and if we had it, universal quantification). While techniques such as freezing [6] or some form of loop checking could be used, we argue that the contraction-free approach is the most natural way to focalize **LB**. The design of such a system, to which we refer as $\mathbf{G4ipf}$, is the main result of the present paper.

As the focusing discipline severely restricts proofs construction, it is paramount to show that we do not lose any proof – in other terms that focusing is complete w.r.t. standard intuitionistic logic. There are in the literature several techniques to prove this, all of them proof-theoretical and none completely satisfactory for our purposes:

1. The permutation-based approach, dating back to Andreoli [1], works by proving inversion properties of asynchronous connectives and postponement properties of synchronous ones, see also [20]. This is very brittle and particularly problematic for contraction-free calculi, as it requires to prove at the same time the admissibility of contraction. In the focusing setting this is far from trivial.
2. One can establish admissibility of the *cut* and of the non-atomic *initial* rule in the focused calculus and then show that all ordinary rules are admissible in the latter using cut. This has been championed in [9]. While a syntactic proof of cut-elimination is an interesting result per se, the sheer number of the judgments involved and hence of the cut reductions (principal, focus, blur, commutative and preserving cuts, in the terminology of the cited paper) makes the well foundedness of the inductive argument very delicate and hard to extend.
3. The so-called "grand-tour through linear logic" strategy of [21]; to show that a refinement of an intuitionistic proof system such as ours is complete, we have to provide an embedding into **LLF** (the canonical focused system for full linear logic) and then show that the latter translation is entailed by Liang and Miller's $1/0$ translation. The trouble here is that contraction-free systems cannot be faithfully encoded in **LLF** [26]. While there are refinements of the latter, namely linear logic with sub-exponentials [28] that may be able to faithfully encode such systems, a "grand-tour" strategy in this context is uncharted territory. Furthermore, sub-exponential encodings of focused systems tend to be very prolix, which makes closing the grand-tour even more problematic.

4. Finally, Miller and Saurin proposed a direct proof of completeness of focusing in linear logic in [27] based on the notion of *focalization graph*. Again, this seems hard to extend to asymmetric calculi such as intuitionism, let alone those contraction-free.

In this paper, which is a revised and extended version of [3], we prove instead completeness adapting the traditional Kripke semantic argument. While this is well-worn in tableaux-like systems, it is the first time that the model-theoretic semantics of focusing has been considered, with the partial exception of [22], where however the calculus under consideration is polarized but not focused. The highlights of our proof are explained in Section 3.

Although stemming from proof-search considerations, focusing has still to make an explicit impact in mainstream theorem proving, keeping in mind, of course, that any sequent calculus implementation will implicitly incorporate on the strategy level some aspects of focusing. There are a few exceptions that we are aware of:

- Inverse-based systems such as Imogen [24] and LIFF [8]: because the inverse method is forward and saturation-based, the issue of contraction does not come into play; in fact the inverse method exhibits different issues w.r.t. termination (namely subsumption) and is not in general geared towards finite failure.
- TAC [7] is a prototype of a family of focused systems for automated (co)inductive theorem proving, including one for **LJF**, see also [29]. Here the emphasis is on the automation of (co)inductive proofs and the objective is to either succeed or quickly fail; hence most care is applied to limit the application of the (co)induction rule by means of *freezing*. Contraction is handled heuristically, by letting the user set a bound for how many times an assumption can be duplicated for each initial goal; once the bound is reached, the system becomes, for all intended purposes, linear.
- Henriksen's [18] presents an analysis of contraction-free classical logic: here contraction has an impact only in the presence of two kinds of disjunctions, namely positive vs. negative. The author shows that contraction can be disposed of by viewing the introduction rule for positive disjunction as a sort of *restart* rule, similar to Gabbay's [17]. The focus rule does not perform any duplication and once the focus is on, say, $A \vee^{+} B$, the introduction rules will maintain focus on B (resp. on A), while adding the discarded alternative A (resp. B) to the set of positive formulas. In case A is negative, it is "positivized", via a *delay*: $\delta^{+}(A) = A \wedge^{+} t^{+}$. This is an interesting approach, but does not seem to help with intuitionistic calculi.

2. The proof system

We consider a standard propositional language based on a enumerable set of atoms, the constant $\bot$ and the connectives $\wedge$, $\vee$ and $\rightarrow$; $\neg A$ is defined as $A \rightarrow \bot$, whereas, for the sake of this paper, we can interpret $\top$ as $\bot \rightarrow \bot$. Our aim is to give a focalized version, that we call $\mathbf{G4ipf}$, of the multi-succedent version of the contraction-free calculus from the VHD lineage. To this end, one starts with a classification of formulas in the (a)synchronous categories. In **LJF** [21], an asynchronous formula has a right invertible rule and a non-invertible left one, and dually for synchronous. The contraction-free approach does not enjoy this symmetry[2]; therefore we just abandon the attempt to try to enforce it. The VHD idea is to

[2] In truth, this symmetry in **LJF** is less compelling, once one considers *positive* conjunction, corresponding to tensor in linear logic; indeed, it is classified as synchronous, notwithstanding the fact that its right rule *is* invertible.

consider the possible shape that the antecedent of an implication can have and provide a specialized left introduction rule. We go further by coupling those left rules with their correspondent right rules, yielding a finer view of implication; in this sense our calculus is reminiscent of Avron's decomposition proof systems [5]. As we shall see shortly, formulas of the form $(A \to B) \to C$ have non-invertible left and right rules, while the rules for $(A \wedge B) \to C$ and $(A \vee B) \to C$ are *both* invertible on the left and on the right. $a \to B$, with a an atom, has a peculiar behavior: the right rule is non-invertible, while the left rule is invertible, provided the left context contains the atom a. This motivates the following, slight unusual, classification of formulas – we discuss the issue of *polarization* of atoms later on in this section.

$$
\begin{array}{rcl}
\text{Async Formula (AF)} & ::= & \bot \mid A \wedge B \mid A \vee B \mid \bot \to B \mid (A \wedge B) \to C \mid (A \vee B) \to C \\
\text{Sync Formula (SF)} & ::= & a \to B \mid (A \to B) \to C \\
\text{AF}^+ & ::= & a \mid \text{AF}
\end{array}
$$

The calculus consists of the following judgments:

- $\mathcal{A}^l; \Theta; \Gamma \Rightarrow \Delta; \Psi; \mathcal{A}^r$: *active* sequent;
- $\mathcal{A}^l; \Theta; A \ll \Psi; \mathcal{A}^r$: *left-focused* sequent;
- $\mathcal{A}^l; \Theta \gg A$: *right-focused* sequent.

$\mathcal{A}^l$ and $\mathcal{A}^r$ denote multisets of atoms, Γ and Δ multisets of arbitrary formulas, while Θ and Ψ are multisets of SF. We adopt most of the convention from [30]; for instance, by Γ, A we mean the multiset union of Γ and $\{A\}$; semicolons are used to separate sets of different kind of formulas. A sequent σ corresponds to the formula $\mathrm{Form}(\sigma)$ defined as follows. Let $\multimap$ be the arrow symbol used in σ (namely, $\multimap$ is one of $\Rightarrow$, $\ll$, $\gg$); then, $\mathrm{Form}(\sigma)$ is the formula $C \to D$, where C is the conjunction of the formulas to the left of $\multimap$ ($\top$ if empty), D is the disjunction of the formulas to the right of $\multimap$ ($\bot$ if empty).

An unusual feature of our calculus is the treatment of atoms and henceforth of the Init axiom rule: not only atoms are singled out in all three judgments, but Init is invoked at the end of the active phase, rather than belonging to the focal one as in standard presentations. Note further that while atoms can be released by the Blur rules, they are *never* focused on. This is motivated again by enforcing termination of bottom-up proof search, in particular to avoid loops induced by focusing and then blurring on atoms.

One may wonder why we take rules as $\bot \to L$ and $\bot \to R$ as primitive; after all, standard presentations of **G4ip** such as [10, 30] do not consider $\bot$ as a possible antecedent of an implication, since the right rule is clearly derivable and the left one would follow from the admissibility of weakening. The same question applies to other right rules such as $\wedge \to R$ and $\vee \to R$. There are several answers to this: first we wish to extend as much as possible each phase, while bundling up all implication right rules in a single rule would entail some pointless phase changes. Further, we believe that it gives the system a pleasant sense of harmony. More importantly, since our calculus is conceived as a decision procedure and one that is proven complete semantically, restricting to the smallest possible core calculus is not a sensible priority. The above rules, in the given versions, could conceivably be shown admissible in a core calculus, provided a series of structural rules and properties hold: weakening, invertibility and postponement of synchronous connectives. Those are surprisingly hard to prove in a system such as ours, while they hold immediately at the semantic level, once completeness is established.

$$\frac{}{\mathcal{A}^l; \Theta; \Gamma, \bot \Rightarrow \Delta; \Psi; \mathcal{A}^r}\ \bot L \qquad \frac{\mathcal{A}^l; \Theta; \Gamma, A, B \Rightarrow \Delta; \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta; \Gamma, A \wedge B \Rightarrow \Delta; \Psi; \mathcal{A}^r}\ \wedge L$$

$$\frac{\mathcal{A}^l; \Theta; \Gamma, A \Rightarrow \Delta; \Psi; \mathcal{A}^r \qquad \mathcal{A}^l; \Theta; \Gamma, B \Rightarrow \Delta; \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta; \Gamma, A \vee B \Rightarrow \Delta; \Psi; \mathcal{A}^r}\ \vee L \qquad \frac{\mathcal{A}^l; \Theta; \Gamma \Rightarrow \Delta; \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta; \Gamma, \bot \to B \Rightarrow \Delta; \Psi; \mathcal{A}^r}\ \bot{\to}L$$

$$\frac{\mathcal{A}^l; \Theta; \Gamma, A \to (B \to C) \Rightarrow \Delta; \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta; \Gamma, (A \wedge B) \to C \Rightarrow \Delta; \Psi; \mathcal{A}^r}\ \wedge{\to}L \qquad \frac{\mathcal{A}^l; \Theta; \Gamma, A \to C, B \to C \Rightarrow \Delta; \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta; \Gamma, (A \vee B) \to C \Rightarrow \Delta; \Psi; \mathcal{A}^r}\ \vee{\to}L$$

$$\frac{\mathcal{A}^l; \Theta, S; \Gamma \Rightarrow \Delta; \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta; \Gamma, S \Rightarrow \Delta; \Psi; \mathcal{A}^r}\ \mathrm{Act}^{\mathrm{l}} \qquad \frac{\mathcal{A}^l, a; \Theta; \Gamma \Rightarrow \Delta; \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta; \Gamma, a \Rightarrow \Delta; \Psi; \mathcal{A}^r}\ \mathrm{Act}^{\mathrm{l}}_{\mathrm{a}}$$

$$\frac{\mathcal{A}^l; \Theta; \Gamma \Rightarrow \Delta; \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta; \Gamma \Rightarrow \bot, \Delta; \Psi; \mathcal{A}^r}\ \bot R \qquad \frac{\mathcal{A}^l; \Theta; \Gamma \Rightarrow A, \Delta; \Psi; \mathcal{A}^r \qquad \mathcal{A}^l; \Theta; \Gamma \Rightarrow B, \Delta; \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta; \Gamma \Rightarrow A \wedge B, \Delta; \Psi; \mathcal{A}^r}\ \wedge R$$

$$\frac{\mathcal{A}^l; \Theta; \Gamma \Rightarrow A, B, \Delta; \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta; \Gamma \Rightarrow A \vee B, \Delta; \Psi; \mathcal{A}^r}\ \vee R \qquad \frac{}{\mathcal{A}^l; \Theta; \Gamma \Rightarrow \bot \to B, \Delta; \Psi; \mathcal{A}^r}\ \bot{\to}R$$

$$\frac{\mathcal{A}^l; \Theta; \Gamma \Rightarrow A \to (B \to C), \Delta; \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta; \Gamma \Rightarrow (A \wedge B) \to C, \Delta; \Psi; \mathcal{A}^r}\ \wedge{\to}R$$

$$\frac{\mathcal{A}^l; \Theta; \Gamma \Rightarrow A \to C, \Delta; \Psi; \mathcal{A}^r \qquad \mathcal{A}^l; \Theta; \Gamma \Rightarrow B \to C, \Delta; \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta; \Gamma \Rightarrow (A \vee B) \to C, \Delta; \Psi; \mathcal{A}^r}\ \vee{\to}R$$

$$\frac{\mathcal{A}^l; \Theta; \Gamma \Rightarrow \Delta; S, \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta; \Gamma \Rightarrow S, \Delta; \Psi; \mathcal{A}^r}\ \mathrm{Act}^{\mathrm{r}} \qquad \frac{\mathcal{A}^l; \Theta; \Gamma \Rightarrow \Delta; \Psi; a, \mathcal{A}^r}{\mathcal{A}^l; \Theta; \Gamma \Rightarrow a, \Delta; \Psi; \mathcal{A}^r}\ \mathrm{Act}^{\mathrm{r}}_{\mathrm{a}}$$

. . . .

$$\frac{\mathcal{A}^l; \Theta; S \ll \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta, S; \cdot \Rightarrow \cdot; \Psi; \mathcal{A}^r}\ \mathrm{F}^{\mathrm{l}} \qquad \frac{\mathcal{A}^l; \Theta \gg S}{\mathcal{A}^l; \Theta; \cdot \Rightarrow \cdot; S, \Psi; \mathcal{A}^r}\ \mathrm{F}^{\mathrm{r}} \qquad \frac{}{\mathcal{A}^l, a; \Theta; \cdot \Rightarrow \cdot; \Psi; a, \mathcal{A}^r}\ \mathrm{Init}$$

$$\frac{\mathcal{A}^l; \Theta; T^+ \Rightarrow \cdot; \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta; T^+ \ll \Psi; \mathcal{A}^r}\ \mathrm{Blur}^{\mathrm{l}} \qquad \frac{\mathcal{A}^l; \Theta; \cdot \Rightarrow T^+; \cdot; \cdot}{\mathcal{A}^l; \Theta \gg T^+}\ \mathrm{Blur}^{\mathrm{r}}$$

. . . .

$$\frac{\mathcal{A}^l, a; \Theta; B \ll \Psi; \mathcal{A}^r}{\mathcal{A}^l, a; \Theta; a \to B \ll \Psi; \mathcal{A}^r}\ \mathrm{at}{\to}L \qquad \frac{\mathcal{A}^l; \Theta; A, B \to C \Rightarrow B; \cdot; \cdot \qquad \mathcal{A}^l; \Theta; C \ll \Psi; \mathcal{A}^l}{\mathcal{A}^l; \Theta; (A \to B) \to C \ll \Psi; \mathcal{A}^r}\ {\to}{\to}L$$

$$\frac{\mathcal{A}^l, a; \Theta \gg B}{\mathcal{A}^l; \Theta \gg a \to B}\ \mathrm{at}{\to}R \qquad \frac{\mathcal{A}^l; \Theta; A \to B \Rightarrow C; \cdot; \cdot}{\mathcal{A}^l; \Theta \gg (A \to B) \to C}\ {\to}{\to}R$$

a is an atom, A, B and C are any formulas, S is a SF, T^+ is an AF^+.

Figure 1. The **G4ipf** calculus

This also applies to all form of cuts, see for example the classification in **LJF** [21]. Finally, the proofs of those properties are rather brittle and therefore hard to extend to larger systems, such as the one we allude to in the following paragraph on polarization or to other very closely connected logics such as Gödel-Dummet's [2,16]. Instead, the semantic approach generalizes much more smoothly. This is not so say that syntactic proofs of structural proprieties are unimportant: indeed they are and very informative too. We simply argue that for our endeavor they do not seem to be appropriate.

We now move to discuss the rules, displayed in Figure 1. As in any focused calculi, proof search alternates between the asynchronous and synchronous phase, depending on the principal formula of the sequent. In the asynchronous phase we eagerly apply the asynchronous rules to an active sequent $\mathcal{A}^l; \Theta; \Gamma \Rightarrow \Delta; \Psi; \mathcal{A}^r$. We arbitrarily select a formula in Γ, Δ:[3] if it is an AF, the formula is decomposed; otherwise, it is moved to one of the synchronous contexts Θ, Ψ, $\mathcal{A}^l$ and $\mathcal{A}^r$ (rule $\mathrm{Act}^{\mathrm{l}}$, $\mathrm{Act}^{\mathrm{l}}_{\mathrm{a}}$, $\mathrm{Act}^{\mathrm{r}}$ or $\mathrm{Act}^{\mathrm{r}}_{\mathrm{a}}$). Once the inner contexts are exhausted, the synchronous phase may start by non deterministically selecting a formula S in Θ, Ψ for focus either on the left (rule F^{l}) or on the right (rule F^{r}). We remark that the main difference between $\mathbf{G4ipf}$ and a calculus such as $\mathbf{LJF}$ is that the rule F^{l} does *not* duplicate the formula selected for focus. This is a crucial point to avoid the generation of branches of infinite length and hence to guarantee the termination of the proof search procedure.

We now come to the focal rules: the left ones are an unsurprising rendering of the corresponding VHD rules, while extending the focusing phase as much as possible. Left focusing terminates, as usual, when an AF^+ formula is encountered, whereby rule $\mathrm{Blur}^{\mathrm{l}}$ returns to the asynchronous phase. Alternatively, a right-focused phase begins by selecting a SF formula from Ψ (rule F^{r}); again, no contraction is performed. According to the cardinality restriction on multi-succedent intuitionistic logic, both right focal rules do not preserve the surrounding right context $\Psi, \mathcal{A}^r$. The $\mathrm{at}\!\to\! R$ rule retains right-focus even though the atom swaps side; however, this is not the case for the $\to\!\to R$, which loses focus and starts a new asynchronous phase. Here there is not much else we can do, as we need to delegate to the active phase the classification of the antecedent $A \to B$. While this is unusual, it is not unprecedented, see the analogous rule in **LJQ*** [11], safe for being applied to any form of implication. In a different but not unrelated context so does the $!R$ rule in [9]. The phase terminates when an AF^+ formula is produced with a call to rule $\mathrm{Blur}^{\mathrm{r}}$.

We end this discussion with an example of a $\mathbf{G4ipf}$-derivation of the formula $\neg\neg(a \vee \neg a)$. Recall that a derivation of such a formula in the standard calculus requires an application of contraction, namely of $\neg(a \vee \neg a)$. The double line corresponds to an asynchronous phase where more than one rule is applied. Building the proof bottom-up, the only backtracking point is the choice of the formula for left-focus in the sequent $\cdot; \neg a, \neg\neg a; \cdot \Rightarrow \cdot; \cdot; \cdot$. If we select $\neg a$ instead of $\neg\neg a$, we get the sequent $\cdot; \neg\neg a; \neg a \ll \cdot; \cdot$ and the construction of the derivation immediately fails.

[3]This choice can be made completely specified by viewing Γ and Δ as lists and choosing an ordering between right and left inversion, as for example in [9].

$$
\dfrac{
\dfrac{
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\dfrac{
\dfrac{}{a;\cdot;\bot \Rightarrow \cdot;\cdot}\,\bot L
}{a;\cdot;\bot \ll \cdot;\cdot}\,\mathrm{Blur}^{\mathrm{l}}
}{a;\cdot;\neg a \ll \cdot;\cdot}\,\mathrm{at}{\to}L
}{a;\neg a;\cdot \Rightarrow \cdot;\cdot;\cdot}\,\mathrm{F}^{\mathrm{l}}
}{\cdot;\neg a;a,\bot \to \bot \Rightarrow \bot;\cdot;\cdot}\,[\bot R, \bot{\to}L, \mathrm{Act}^{\mathrm{l}}_{\mathrm{a}}]
\qquad
\dfrac{
\dfrac{}{\cdot;\neg a;\bot \Rightarrow \cdot;\cdot;\cdot}\,\bot L
}{\cdot;\neg a;\bot \ll \cdot;\cdot}\,\mathrm{Blur}^{\mathrm{l}}
}{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\cdot;\neg a;\neg\neg a \ll \cdot;\cdot}{\cdot;\neg a,\neg\neg a;\cdot \Rightarrow \cdot;\cdot;\cdot}\,\mathrm{F}^{\mathrm{l}}
}{\cdot;\cdot;\neg(a \vee \neg a) \Rightarrow \bot;\cdot;\cdot}\,[\bot R, \vee{\to}L, \mathrm{Act}^{\mathrm{l}} \times 2]
}{\cdot;\cdot \gg \neg\neg(a \vee \neg a)}\,{\to\to}R
}{\cdot;\cdot;\cdot \Rightarrow \cdot;\neg\neg(a \vee \neg a);\cdot}\,\mathrm{F}^{\mathrm{r}}
}{\cdot;\cdot;\cdot \Rightarrow \neg\neg(a \vee \neg a);\cdot;\cdot}\,\mathrm{Act}^{\mathrm{r}}
}\,{\to\to}L
$$

Polarization Almost every modern treatment of focusing, see e.g. [21], extends the (a)synchronous classification of connectives to *atoms*, assigning them a *bias* or *polarity*. Different polarizations of atoms do not affect provability, but do influence significantly the *shape* of the derivation, allowing one to informally[4] characterize forward and backward reasoning via respectively positive and negative bias assignments. Unfortunately the contraction-free approach seems essentially forward and negative bias does not behave as expected. In a putative calculus with atomic bias, the new rules, apart from the Init ones, would be the left rules for atomic implication, where $\mathrm{at} \to L$ would be now reserved to positive atoms, let us call it $\to \mathrm{at}^+$, while for negative ones we would have in rule $\to \mathrm{at}^-$ below a variant of $\to\to L$, namely:

$$
\dfrac{\mathcal{A}^l;\Theta;\cdot \Rightarrow \cdot;\cdot;n \qquad \mathcal{A}^l;\Theta;B \ll \Psi;\mathcal{A}^r}{\mathcal{A}^l;\Theta;n \to B \ll \Psi;\mathcal{A}^r}\to\mathrm{at}^-
\qquad\qquad
\dfrac{\mathcal{A}^l,p;\Theta;B \ll \Psi;\mathcal{A}^r}{\mathcal{A}^l,p;\Theta;p \to B \ll \Psi;\mathcal{A}^r}\to\mathrm{at}^+
$$

where $n(p)$ is a negative (positive) atom. However, in the absence of contraction, these rules do not yield a complete calculus. For instance, consider the provable sequent $\sigma = \cdot;n \to p,(n \to p) \to n;\cdot \Rightarrow \cdot;\cdot;p$, whose informal backward chaining proof requires two usage of $n \to p$. The only rule applicable to σ is F^{l} and if we select $n \to p$

$$
\dfrac{\cdot;(n \to p) \to n;\cdot \Rightarrow \cdot;\cdot;n \qquad \begin{array}{c}\vdots\\ \cdot;(n \to p) \to n;p \ll \cdot;p\end{array}}{\cdot;(n \to p) \to n;n \to p \ll \cdot;p}\to\mathrm{at}^-
$$

The left premise is not provable. On the other hand, if we focus on $(n \to p) \to n$ we get:

$$
\dfrac{\begin{array}{c}\vdots\\ \cdot;n \to p;n,p \to n \Rightarrow p;\cdot;\cdot\end{array} \qquad \cdot;n \to p;n \ll \cdot;p}{\cdot;n \to p;(n \to p) \to n \ll \cdot;p}\to\to L
$$

Here the right premise is not provable, because we do not permit blurring focus on negative atoms. If we do allow that, it is not clear that the calculus does properly capture the notion of "backward chaining" anymore.

[4]The only formal characterization of this idea that we are aware of is for the Horn clause fragment, using SLD resolution for backward and hyper-resolution for forward chaining [9].

On the other hand it seems quite possible to distinguish between a positive and negative conjunction, as commonly done, see e.g. [21]. We can go further: since we are focusing a **LB**-like calculus, we can easily accommodate positive and negative *disjunctions*, a distinction that is moot in intuitionistic single succedent systems. The introduction of negative conjunction has an interesting interaction with contraction: as a negative connective, its left-focal rule drops one of the conjuncts. This would not be an issue were we to allow the F^{l} rule to duplicate the focused-on formula, but we do not, of course. A solution is to follow on Henriksen's idea [18] and consider a rule such as:

$$\frac{\mathcal{A}^l; \Theta, \delta(A_j); A_i \ll \Psi; \mathcal{A}^r}{\mathcal{A}^l; \Theta; A_0 \wedge^- A_1 \ll \Psi; \mathcal{A}^r} \wedge^-_{i,j} L$$

where we keep a copy of the discarded conjuncts appropriately synchronized with a delay. As the completeness proof of such an extension becomes somewhat more complex, we leave that to future work.

Termination. To show that $\mathbf{G4ipf}$ is terminating we define, as usual, a well-founded relation $\prec$ such that if σ is the conclusion of a rule r of $\mathbf{G4ipf}$ and σ' any of the premises of r, then $\sigma' \prec \sigma$. As a consequence, branches of infinite length cannot be generated during proof search. We start by assigning *weights* to formulas and sequents; the assignment is standard and follows [30]:

$$\begin{aligned}
\mathrm{wg}(a) \;=\; \mathrm{wg}(\bot) \;&=\; 2\\
\mathrm{wg}(A \wedge B) \;&=\; \mathrm{wg}(A) + (\mathrm{wg}(A) \cdot \mathrm{wg}(B))\\
\mathrm{wg}(A \vee B) \;&=\; 1 + \mathrm{wg}(A) + \mathrm{wg}(B)\\
\mathrm{wg}(A \to B) \;&=\; 1 + (\mathrm{wg}(A) \cdot \mathrm{wg}(B))
\end{aligned}$$

One can double check that:

- $\mathrm{wg}(A \to (B \to C)) \;<\; \mathrm{wg}((A \wedge B) \to C)$;
- $\mathrm{wg}(A \to C) + \mathrm{wg}(B \to C) \;<\; \mathrm{wg}((A \vee B) \to C)$;
- $\mathrm{wg}(A) + \mathrm{wg}(B \to C) + \mathrm{wg}(C) \;<\; \mathrm{wg}((A \to B) \to C)$.

The above properties suffice to prove that the standard (unfocused) calculus $\mathbf{G4ip}$ terminates [10]. Note instead that in $\mathbf{G4ipf}$ we cannot use the weight of the whole sequent as a measure, since in the rules that govern phase changes and that move formulas between contexts the conclusion and the premise have the same weight. Thus our well-founded relation will be computed from two relations $\prec_s$ and $\prec_d$, the first relating sequents in the same phase, the other relating sequents obtained by the application of a rule that starts or ends a synchronous phase. More precisely, $\sigma_1 \prec_s \sigma_2$ (respectively $\sigma_1 \prec_d \sigma_2$) holds iff there exists a rule r of $\mathbf{G4ipf}$ such that σ_2 is the conclusion, σ_1 is any of the premises of r, and σ_1 and σ_2 are of the same (respectively different) kind, i.e., they belong to the same (respectively different) judgment. Take $\prec$ to be the transitive closure of the relation $\prec_s \cup \prec_d$.

Lemma 2.1. $\prec_s$ is a well-founded order.

Proof:
We prove that every $\prec_s$-chain $\mathcal{C}$ of the form $\cdots \prec_s \sigma_3 \prec_s \sigma_2 \prec_s \sigma_1$ is finite. By definition all the sequents σ_n in $\mathcal{C}$ are of the same kind. If all the σ_n are left/right-focused, the lemma follows from the

fact that along $\mathcal{C}$ the weight of the formula under focus decreases. Let us assume that, for every $n \geq 1$, $\sigma_n = \mathcal{A}^l_n; \Theta_n; \Gamma_n \Rightarrow \Delta_n; \Psi_n; \mathcal{A}^r_n$ and let r_n be the rule of **G4ipf** such that σ_n is the conclusion of r_n and σ_{n+1} a premise of r_n; we use the lexicographic ordering on $\langle \mathrm{wg}(\sigma_n), \mathrm{wg}(\Gamma_n, \Delta_n)\rangle$. If r_n is not an Act rule, a formula in Γ_n, Δ_n is decomposed, hence $\mathrm{wg}(\sigma_{n+1}) < \mathrm{wg}(\sigma_n)$. If r_n is an Act rule, a formula in Γ_n, Δ_n is moved outward, hence $\mathrm{wg}(\sigma_{n+1}) = \mathrm{wg}(\sigma_n)$ and $\mathrm{wg}(\Gamma_{n+1}, \Delta_{n+1}) < \mathrm{wg}(\Gamma_n, \Delta_n)$. Thus, $\mathcal{C}$ is finite. □

The following lemma ensures that two active sequents σ_a and σ_b immediately after and before a synchronous phase have decreasing weights.

Lemma 2.2. Let σ_a and σ_b be two active sequents and $\sigma_1, \ldots, \sigma_n$ be $n \geq 1$ focused sequents such that $\sigma_a \prec_d \sigma_1 \prec_s \cdots \prec_s \sigma_n \prec_d \sigma_b$. Then $\mathrm{wg}(\sigma_a) < \mathrm{wg}(\sigma_b)$.

Proof:
If σ_a and σ_b mark a right-focused phase, at least one application of rule $\mathrm{at}{\rightarrow}R$ or $\rightarrow\!\!\rightarrow R$ is performed, hence $\mathrm{wg}(\sigma_a) < \mathrm{wg}(\sigma_b)$. W.r.t. a left-focus phase, first observe that the chain $\sigma_a \prec_d \sigma_1 \prec_d \sigma_b$, corresponding to an application of F^{l} immediately followed by $\mathrm{Blur}^{\mathrm{l}}$ is forbidden, since the formula selected by F^{l} must be a (non-atomic) SF, whereas $\mathrm{Blur}^{\mathrm{l}}$ can only release an AF^+. Accordingly, at least an application of $\mathrm{at}{\rightarrow}L$ or $\rightarrow\!\!\rightarrow L$ must occur. □

Proposition 2.3. $\prec$ is a well-founded order.

Proof:
Assume, by absurd, that there exists an infinite $\prec$-chain $\mathcal{C}$ of sequents σ_n ($n \geq 1$) such that $\sigma_{n+1} \prec \sigma_n$ for every $n \geq 1$. We have $\mathrm{wg}(\sigma_{n+1}) \leq \mathrm{wg}(\sigma_n)$ for every $n \geq 1$. By Lemma 2.1, the relation $\prec_s$ is well-founded, hence $\mathcal{C}$ contains infinitely many occurrences of $\prec_d$. By Lemma 2.2, from $\mathcal{C}$ we can extract an infinite sequence of active sequents σ'_n such that $\mathrm{wg}(\sigma'_{n+1}) < \mathrm{wg}(\sigma'_n)$ for every $n \geq 1$, a contradiction. □

3. Soundness and Completeness

We take as our notion of Kripke model a structure $\mathcal{K} = \langle P, \leq, \rho, V\rangle$, where $\langle P, \leq, \rho\rangle$ is a finite poset with minimum element ρ and V is a function mapping every $\alpha \in P$ to a subset of atoms such that $\alpha \leq \beta$ implies $V(\alpha) \subseteq V(\beta)$. We write $\alpha < \beta$ to mean $\alpha \leq \beta$ and $\alpha \neq \beta$. The *forcing relation* $\mathcal{K}, \alpha \Vdash H$ (α forces formula H in $\mathcal{K}$) is defined as follows:

- $\mathcal{K}, \alpha \nVdash \bot$;
- for every atom a, $\mathcal{K}, \alpha \Vdash a$ iff $a \in V(\alpha)$;
- $\mathcal{K}, \alpha \Vdash A \wedge B$ iff $\mathcal{K}, \alpha \Vdash A$ and $\mathcal{K}, \alpha \Vdash B$;
- $\mathcal{K}, \alpha \Vdash A \vee B$ iff $\mathcal{K}, \alpha \Vdash A$ or $\mathcal{K}, \alpha \Vdash B$;
- $\mathcal{K}, \alpha \Vdash A \rightarrow B$ iff, for every $\beta \in P$, $\alpha \leq \beta$ and $\mathcal{K}, \beta \Vdash A$ imply $\mathcal{K}, \beta \Vdash B$.

As well-known, *monotonicity* holds for arbitrary formulas: $\mathcal{K}, \alpha \Vdash A$ and $\alpha \leq \beta$ imply $\mathcal{K}, \beta \Vdash A$. We define a formula A to be *valid in* $\mathcal{K}$ iff $\mathcal{K}, \rho \Vdash A$.

Let $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ be a Kripke model, $\alpha \in P$, σ a sequent and $\mathrm{Form}(\sigma) = C \to D$; we say that $\mathcal{K}$ *realizes* σ at α, written $\mathcal{K}, \alpha \rhd \sigma$, iff $\mathcal{K}, \alpha \Vdash C$ and $\mathcal{K}, \alpha \nVdash D$ (hence, $\mathcal{K}, \alpha \nVdash \mathrm{Form}(\sigma)$). A sequent σ is *realizable* if there exists a model $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ such that $\mathcal{K}, \rho \rhd \sigma$; in this case we say that $\mathcal{K}$ is a *model* of σ. It immediately follows that σ is realizable iff the formula $\mathrm{Form}(\sigma)$ is *not* intuitionistically valid.

A rule r is *sound* iff at least one of its premises is realizable, provided that the conclusion of r is realizable. We can easily prove that the rules of $\mathbf{G4ipf}$ are sound, yielding:

Theorem 3.1. (Soundness)
If σ is provable in $\mathbf{G4ipf}$ then σ is not realizable.

We can now move to the more interesting direction, $\mathbf{G4ipf}$'s completeness; the plan is to show that if proof search for a sequent σ fails we can build a model $\mathcal{K}$ of σ. Henceforth, by *unprovable* we mean 'not provable in $\mathbf{G4ipf}$'. In standard calculi for intuitionistic logic, whenever a sequent σ is not provable, we can build a model of σ out of the open branches of the derivations generated during proof search; this shows that a not provable sequent is realizable. This strategy does not work in $\mathbf{G4ipf}$, since in a left-focal phase we can find unprovable left-focused sequents that are *not* realizable (e.g., the sequent $b; \cdot; a \to b \ll \cdot; b$).

We begin by introducing a classification of a *failed* left-focal phase: such a phase can fail either because the application of Blur^l yields an unprovable active sequent, or because at some point the formula under focus is $a \to B$ and the application of $\mathrm{at} \to L$ is not allowed (a is not in $\mathcal{A}^l$) or, finally, because the focus is on $(A \to B) \to C$ and the left hand-side premise of $\to\!\!\to L$ is unprovable.

Definition 3.2. Let σ be the left-focused sequent $\mathcal{A}^l; \Theta; H \ll \Psi; \mathcal{A}^r$:

- σ is *strongly unprovable* iff one of the following conditions holds:
    1. H is an AF^+ and the sequent $\mathcal{A}^l; \Theta; H \Rightarrow \cdot; \Psi; \mathcal{A}^r$ is unprovable;
    2. $H = A \to B$ and $\mathcal{A}^l; \Theta; B \ll \Psi; \mathcal{A}^r$ is strongly unprovable.
- σ is *at-unprovable w.r.t.* $a \to B$ iff one of the following conditions holds:
    1. $H = a \to B$ and $a \notin \mathcal{A}^l$;
    2. $H = K \to L$ and $\mathcal{A}^l; \Theta; L \ll \Psi; \mathcal{A}^r$ is at-unprovable w.r.t. $a \to B$.

  σ is *at-unprovable* if, for some $a \to B$, σ is at-unprovable w.r.t. $a \to B$.
- σ is *$\to$-unprovable w.r.t.* $(A \to B) \to C$ iff one of the following conditions holds:
    1. $H = (A \to B) \to C$ and $\mathcal{A}^l; \Theta; A, B \to C \Rightarrow B; \cdot; \cdot$ is unprovable;
    2. $H = K \to L$ and $\mathcal{A}^l; \Theta; L \ll \Psi; \mathcal{A}^r$ is $\to$-unprovable w.r.t. $(A \to B) \to C$.

  σ is *$\to$-unprovable* if, for some $(A \to B) \to C$, σ is $\to$-unprovable w.r.t. $(A \to B) \to C$.

The above notions may overlap. For instance, let $\sigma = \cdot; \cdot; a_1 \to (a_2 \to a_3) \to a_4 \to a_5 \ll \cdot; \cdot$; then:

- σ is strongly unprovable;
- σ is at-unprovable w.r.t. $a_1 \to (a_2 \to a_3) \to a_4 \to a_5$ and w.r.t. $a_4 \to a_5$;
- σ is $\to$-unprovable w.r.t. $(a_2 \to a_3) \to a_4 \to a_5$.

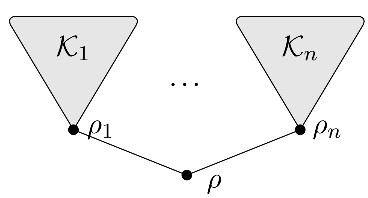
A left-focused unprovable sequent matches at least one of the above definitions, as the next lemma shows.

Lemma 3.3. Let $\sigma = \mathcal{A}^l; \Theta; H \ll \Psi; \mathcal{A}^r$ be an unprovable sequent. Then σ is strongly unprovable or at-unprovable or $\rightarrow$-unprovable.

Proof:
By induction on $\prec$ and case analysis on H. Let H be an AF^+. Since σ is unprovable, so is $\mathcal{A}^l; \Theta; H \Rightarrow \cdot; \Psi; \mathcal{A}^r$ (Blur^l). Hence by definition σ is strongly unprovable. Let $H = a \rightarrow B$. If $a \notin \mathcal{A}^l$ then σ is at-unprovable w.r.t. $a \rightarrow B$. Otherwise, let $a \in \mathcal{A}^l$; by rule $\mathrm{at} \rightarrow L$, the sequent $\sigma' = \mathcal{A}^l; \Theta; B \ll \Psi; \mathcal{A}^r$ is unprovable. Since $\sigma' \prec \sigma$, by IH σ' is strongly unprovable or at-unprovable or $\rightarrow$-unprovable. According to the case, it follows that σ is strongly unprovable or at-unprovable or $\rightarrow$-unprovable. Let $H = (A \rightarrow B) \rightarrow C$. If $\sigma' = \mathcal{A}^l; \Theta; A, B \rightarrow C \Rightarrow B; \cdot; \cdot$ is unprovable, then σ is $\rightarrow$-unprovable w.r.t. $(A \rightarrow B) \rightarrow C$. Otherwise, let σ' be provable; by rule $\rightarrow\rightarrow L$, $\sigma'' = \mathcal{A}^l; \Theta; C \ll \Psi; \mathcal{A}^r$ is unprovable. Since $\sigma'' \prec \sigma$, we can apply the IH to σ'', and this proves the lemma for σ. $\Box$

We introduce the *gluing* model operation that we use for our counter-model construction. Let $\mathcal{S} = \{\mathcal{K}_1, \ldots \mathcal{K}_n\}$ be a (possibly empty) set of models $\mathcal{K}_i = \langle P_i, \leq_i, \rho_i, V_i \rangle$ $(1 \leq i \leq n)$, At a set of atoms such that, for every $1 \leq i \leq n$, $\mathrm{At} \subseteq V_i(\rho_i)$; without loss of generality, we can assume that the sets P_i are pairwise disjoint. By $\mathrm{Model}(\mathrm{At}, \mathcal{S})$ we denote the following Kripke model $\mathcal{K} = \langle P, \leq, \rho, V \rangle$:

- If $\mathcal{S}$ is empty, then $\mathcal{K}$ is the Kripke model consisting of the singleton world ρ and $V(\rho) = \mathrm{At}$.
- Let $n \geq 1$. Then (see the picture below):
  - ρ is new (namely, $\rho \notin \bigcup_{i \in \{1,\ldots,n\}} P_i$) and $P = \{\rho\} \cup \bigcup_{i \in \{1,\ldots,n\}} P_i$;
  - $\leq = \{ (\rho, \alpha) \mid \alpha \in P \} \cup \bigcup_{i \in \{1,\ldots,n\}} \leq_i$;
  - $V(\rho) = \mathrm{At}$ and, for $1 \leq i \leq n$ and $\alpha \in P_i$, $V(\alpha) = V_i(\alpha)$.



It is easy to check that $\mathcal{K}$ is a well-defined Kripke model and that for every $1 \leq i \leq n$, $\alpha \in P_i$ and formula A, it holds that $\mathcal{K}, \alpha \Vdash A$ iff $\mathcal{K}_i, \alpha \Vdash A$.

Let $\sigma = \mathcal{A}^l; \Theta; \cdot \Rightarrow \cdot; \Psi; \mathcal{A}^r$ and $H \in \Theta$; by $\sigma_H^{\ll}$ we denote the sequent $\mathcal{A}^l; \Theta \setminus \{H\}; H \ll \Psi; \mathcal{A}^r$ obtained at the beginning of a left-focal phase starting from σ with focus on H (see rule F^l). The crucial point in the counter-model construction is the definition of a model $\mathcal{K}$ of σ in the case that, for every $H \in \Theta$, $\sigma_H^{\ll}$ is at-unprovable or $\rightarrow$-unprovable. To build $\mathcal{K}$, for every $A \rightarrow B \in \Psi$ we need a model of $\mathcal{A}^l; \Theta; A \Rightarrow B; \cdot; \cdot$, (as in the standard construction), which witnesses that $A \rightarrow B$ is not forced in the root ρ of $\mathcal{K}$. Pick a $H \in \Theta$ such that $\sigma_H^{\ll}$ is not at-unprovable and assume that $\sigma_H^{\ll}$ is $\rightarrow$-unprovable w.r.t. $(A \rightarrow B) \rightarrow C$: to guarantee that H is forced in ρ, we need a model of $\mathcal{A}^l; \Theta \setminus \{H\}; A, B \rightarrow C \Rightarrow B; \cdot; \cdot$. In the next lemma we prove that we can build a model of σ, by gluing all these models together.

Lemma 3.4. Let $\sigma = \mathcal{A}^l; \Theta; \cdot \Rightarrow \cdot; \Psi; \mathcal{A}^r$ be an unprovable sequent such that for every $H \in \Theta$ the sequent $\sigma_H^{\ll}$ is at-unprovable or $\rightarrow$-unprovable. Let Θ_0 be the set of formulas $H \in \Theta$ such that $\sigma_H^{\ll}$ is at-unprovable and let $\Theta_1 = \Theta \setminus \Theta_0$. Define $\mathcal{S}$ to be a (possibly empty) set of models satisfying the following conditions:

- (i) For every $H \in \Theta_1$, let $(A \rightarrow B) \rightarrow C$ be such that $\sigma_H^{\ll}$ is $\rightarrow$-unprovable w.r.t. $(A \rightarrow B) \rightarrow C$; then $\mathcal{S}$ contains a model of the sequent $\mathcal{A}^l; \Theta \setminus \{H\}; A, B \rightarrow C \Rightarrow B; \cdot; \cdot$.
- (ii) For every $A \rightarrow B \in \Psi$, $\mathcal{S}$ contains a model of the sequent $\mathcal{A}^l; \Theta; A \Rightarrow B; \cdot; \cdot$.
- (iii) Every model of $\mathcal{S}$ is of type (i) or (ii).

Then $\mathrm{Model}(\mathcal{A}^l, \mathcal{S})$ is a model of σ.

Proof:
Let $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ be the model $\mathrm{Model}(\mathcal{A}^l, \mathcal{S})$. By definition, $V(\rho) = \mathcal{A}^l$. To prove that $\mathcal{K}$ is a model of σ, we have to show the following facts:

- (F1) $\mathcal{K}, \rho \Vdash a$ for every $a \in \mathcal{A}^l$, and $\mathcal{K}, \rho \nVdash a'$ for every $a' \in \mathcal{A}^r$;
- (F2) $\mathcal{K}, \rho \Vdash H$ for every $H \in \Theta$;
- (F3) $\mathcal{K}, \rho \nVdash A \rightarrow B$ for every $A \rightarrow B \in \Psi$.

Point (F1) immediately follows by the definition of V and the fact that the sets $\mathcal{A}^l$ and $\mathcal{A}^r$ are disjoint. We prove (F2) and (F3) by a case analysis on $\mathcal{S}$. Suppose $\mathcal{S}$ is empty; then $\mathcal{K}$ only contains the world ρ, and moreover $\Theta = \Theta_0$ and $\Psi = \emptyset$. Let $H \in \Theta$ and let $\sigma_H^{\ll}$ be at-unprovable w.r.t. $a \rightarrow B$. Since $\mathcal{K}, \rho \nVdash a$ (indeed, $a \notin \mathcal{A}^l$) and ρ has no proper successors in $\mathcal{K}$, it holds that $\mathcal{K}, \rho \Vdash a \rightarrow B$ and, having H the form $H_1 \rightarrow \cdots \rightarrow a \rightarrow B$, this implies (F2). Point (F3) is trivial.
Suppose now that $\mathcal{S}$ contains the models $\mathcal{K}_1 = \langle P_1, \leq_1, \rho_1, V_1 \rangle, \ldots, \mathcal{K}_n = \langle P_n, \leq_n, \rho_n, V_n \rangle$ $(n \geq 1)$. To show that (F2) holds, pick $H \in \Theta_0$ and let $\sigma_H^{\ll}$ be at-unprovable w.r.t. $a \rightarrow B$; then, H has the form $H_1 \rightarrow \cdots \rightarrow a \rightarrow B$, where $a \notin \mathcal{A}^l$. We note that $\mathcal{K}_i, \rho_i \Vdash H$, for every $1 \leq i \leq n$; indeed, by the hypothesis (i)– (iii), $\mathcal{K}_i$ is a model of a sequent of the form $\mathcal{A}^l; \Theta'; \Gamma' \Rightarrow \Delta'; \cdot; \cdot$ such that $H \in \Theta'$. It follows that $\mathcal{K}_i, \rho_i \Vdash H$, for every $1 \leq i \leq n$; hence $\mathcal{K}, \rho_i \Vdash H$. By definition of V, we have $\mathcal{K}, \rho \nVdash a$; we conclude $\mathcal{K}, \rho \Vdash H$. Let $H \in \Theta_1$ and let $\sigma_H^{\ll}$ be $\rightarrow$-unprovable w.r.t. $(A \rightarrow B) \rightarrow C$ (see hypothesis (i)). By definition, we can set $H = H_1 \rightarrow \cdots \rightarrow (A \rightarrow B) \rightarrow C$ and, by (i), $\mathcal{S}$ contains a model $\mathcal{K}_j$ of $\mathcal{A}^l; \Theta \setminus \{H\}; A, B \rightarrow C \Rightarrow B; \cdot;$. This implies that:

- (P1) $\mathcal{K}_j, \rho_j \Vdash A$
- (P2) $\mathcal{K}_j, \rho_j \Vdash B \rightarrow C$
- (P3) $\mathcal{K}_j, \rho_j \nVdash B$

(P1) and (P2) entails that $\mathcal{K}_j, \rho_j \Vdash (A \rightarrow B) \rightarrow C$, which implies $\mathcal{K}_j, \rho_j \Vdash H$. Moreover, if $l \in \{1, \ldots, n\}$ and $l \neq j$, then by (i)– (iii) $\mathcal{K}_l$ is a model of a sequent $\mathcal{A}^l; \Theta'; \Gamma' \Rightarrow \Delta'; \cdot; \cdot$ such that $H \in \Theta'$, hence $\mathcal{K}_l, \rho_l \Vdash H$. Thus, for every $1 \leq i \leq n$, it holds that $\mathcal{K}_i, \rho_i \Vdash H$, which implies $\mathcal{K}, \rho_i \Vdash H$. By (P1) and (P3) and the fact that $\rho < \rho_j$ in $\mathcal{K}$, we get $\mathcal{K}, \rho \nVdash A \rightarrow B$; we conclude $\mathcal{K}, \rho \Vdash H$, and this proves (F2). The proof of (F3) follows by the fact that, for every $A \rightarrow B \in \Psi$, $\mathcal{S}$ contains a model $\mathcal{K}_j$ of $\mathcal{A}^l; \Theta; A \Rightarrow B; \cdot; \cdot$ (see hypothesis (ii)) and ρ_j is a successor of ρ in $\mathcal{K}$. $\square$

We are now ready for:

Proposition 3.5. (Completeness)

(i) If $\sigma = \mathcal{A}^l; \Theta; \Gamma \Rightarrow \Delta; \Psi; \mathcal{A}^r$ is unprovable, then σ is realizable.

(ii) If $\sigma = \mathcal{A}^l; \Theta; H \ll \Psi; \mathcal{A}^r$ is strongly unprovable, then σ is realizable.

(iii) If $\sigma = \mathcal{A}^l; \Theta \gg H$ is unprovable, then σ is realizable.

Proof:

We prove (i),(ii) and (iii) by induction on $\prec$. We start with (i). If Γ, Δ is not empty, (i) easily follows by IH(i). For instance, let $\sigma = \mathcal{A}^l; \Theta; \Gamma, A \vee B \Rightarrow \Delta; \Psi; \mathcal{A}^r$. By definition of rule $\vee L$, one of the sequents $\sigma_1 = \mathcal{A}^l; \Theta; \Gamma, A \Rightarrow \Delta; \Psi; \mathcal{A}^r$ and $\sigma_2 = \mathcal{A}^l; \Theta; \Gamma, B \Rightarrow \Delta; \Psi; \mathcal{A}^r$ is unprovable. Let σ_k be unprovable ($k \in \{1, 2\}$). Since $\sigma_k \prec \sigma$, by IH(i) there exists a model $\mathcal{K}$ of σ_k. Since $\mathcal{K}$ is a model of σ, σ is realizable. Let $\sigma = \mathcal{A}^l; \Theta; \cdot \Rightarrow \cdot; \Psi; \mathcal{A}^r$. We distinguish two cases (C1) and (C2).

(C1) There exists a formula $H \in \Theta$ such that $\sigma_H^{\ll}$ is strongly unprovable.

Since $\sigma_H^{\ll} \prec \sigma$, by IH(ii) there exists a model $\mathcal{K}$ of $\sigma_H^{\ll}$. Since $\mathrm{Form}(\sigma_H^{\ll}) = \mathrm{Form}(\sigma)$, $\mathcal{K}$ is a model of σ, hence σ is realizable.

(C2) For every $H \in \Theta$, the sequent $\sigma_H^{\ll}$ is not strongly unprovable.

We build a model of σ by applying Lemma 3.4. First, we check that the hypothesis of Lemma 3.4 are satisfied. By (C2) and Lemma 3.3, it follows that, for every $H \in \Theta$, $\sigma_H^{\ll}$ is at-unprovable or $\to$-unprovable. To define the set of models $\mathcal{S}$ satisfying the hypothesis Lemma 3.3 we note that:

(a) Let $H \in \Theta_1$ and $(A \to B) \to C$ be such that $\sigma_H^{\ll}$ is $\to$-unprovable w.r.t. $(A \to B) \to C$. Then the sequent $\sigma' = \mathcal{A}^l; \Theta \setminus \{H\}; A, B \to C \Rightarrow B; \cdot; \cdot$ is unprovable. Since $\sigma' \prec \sigma$, by IH(i) there exists a model of σ'.

(b) Let $S \in \Psi$. If $S = a \to B$, then the sequent $\sigma' = \mathcal{A}^l, a; \Theta \gg B$ is unprovable (otherwise, by applying rules F^{r} and $\mathrm{at}{\to}R$, σ would be provable). Since $\sigma' \prec \sigma$, by IH(iii) there exists a model $\mathcal{K}'$ of σ', hence $\mathcal{K}'$ is a model of $\mathcal{A}^l; \Theta; a \Rightarrow B; \cdot; \cdot$. Similarly, let $S = (A \to B) \to C$; since $\sigma' = \mathcal{A}^l; \Theta; A \to B \Rightarrow C; \cdot; \cdot$ is unprovable, by IH(i) there exists a model of σ'.

Let $\mathcal{S}$ be the (possibly empty) set of models mentioned in (a) and (b); by Lemma 3.4, $\mathrm{Model}(\mathrm{At}^l, \mathcal{S})$ is a model of σ, hence σ is realizable. This concludes the proof of (i).

To prove (ii), assume that $\sigma = \mathcal{A}^l; \Theta; H \ll \Psi; \mathcal{A}^r$ is strongly unprovable. If H is an AF^+, the sequent $\sigma' = \mathcal{A}^l; \Theta; H \Rightarrow \cdot; \Psi; \mathcal{A}^r$ is unprovable. Since $\sigma' \prec \sigma$, by IH(i) there exists a model $\mathcal{K}$ of σ'; hence, $\mathcal{K}$ is a model of σ and σ is realizable. Let $H = A \to B$; then, $\sigma' = \mathcal{A}^l; \Theta; B \ll \Psi; \mathcal{A}^r$ is strongly unprovable. Since $\sigma' \prec \sigma$, by IH(ii) there exists a model $\mathcal{K}$ of σ'. Since $\mathcal{K}, \rho \Vdash B$, it follows that $\mathcal{K}, \rho \Vdash A \to B$, hence $\mathcal{K}$ is a model of σ. This concludes the proof of (ii).

Finally, let $\sigma = \mathcal{A}^l; \Theta \gg H$ be unprovable. If H is an AF^+, then the sequent $\sigma' = \mathcal{A}^l; \Theta; \cdot \Rightarrow H; \cdot; \cdot$ is unprovable. Since $\sigma' \prec \sigma$, by IH(i) there exists a model $\mathcal{K}$ of σ' and, since $\mathcal{K}$ is a model of σ, σ is realizable. Let $H = a \to B$; then, $\sigma' = \mathcal{A}^l, a; \Theta \gg B$ is unprovable. Since $\sigma' \prec \sigma$, by IH(iii) there exists a model $\mathcal{K}$ of σ'. Since $\mathcal{K}, \rho \Vdash a$ and $\mathcal{K}, \rho \nVdash B$, it follows that $\mathcal{K}, \rho \nVdash a \to B$, hence $\mathcal{K}$ is a model of σ. The remaining case $H = (A \to B) \to C$ is similar. □

The above proof shows how to build a model of an unprovable sequent (see in particular points (a) and (b)). By soundness and completeness of $\mathbf{G4ipf}$, a sequent σ is provable in $\mathbf{G4ipf}$ iff σ is not realizable. If $\sigma_A = \cdot; \cdot; \cdot \Rightarrow A; \cdot; \cdot$, then $\mathrm{Form}(\sigma_A) = \top \to A$: σ_A is realizable iff A is not intuitionistically valid. By soundness and completeness of $\mathbf{G4ipf}$, $A \in \mathbf{Int}$ iff σ_A is provable in $\mathbf{G4ipf}$.

4. Conclusions

We have presented **G4ipf**, a contraction-free focused multi-succedent sequent calculus for propositional intuitionistic logic, following ideas coming from Vorob'ev, Hudelmeier and Dyckhoff. We have proven completeness of the calculus semantically and argued that this offers a viable alternative to the usual syntactical techniques that do not seem to be easily applicable in our case. This paper is just a beginning of our investigation of focusing: it is commonly believed, in fact, that every "reasonable" sequent calculus has a natural focused version. We aim to test this "universality" hypothesis further by investigating its applicability to a rather peculiar logic, Gödel-Dummett's [2, 16], which is well-known to lead a double life as a super-intuitionistic (but not constructive) and quintessential fuzzy logic. We plan to investigate further the issue of polarization and to extend our completeness result to a logic with (a)synchronous conjunction and disjunction.

Another open question is *counter-example* search in focused systems: considering that such calculi restrict the shape of derivations, what kind of counter models do they yield, upon failure? How do they compare to calculi such as [4] or the calculus [13] designed to yield models of *minimal* depth?

Finally, there seems to be a connection between contraction-free calculi and Gabbay's restart rule [17], a technique to make goal oriented provability with diminishing resources complete for intuitionistic provability. Focusing could be the key to understand this.

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