Remember

The deductive system you will be learning is a Fitch-style deductive system, named $\mathcal{F}$. The computer application that assists you in constructing proofs in $\mathcal{F}$ is therefore called Fitch. If you write out your proofs on paper, you are using the system $\mathcal{F}$, but not the program Fitch.

Exercises

2.15 If you skipped the **You try it** sections, go back and do them now. Submit the files Proof Identity 1 and Proof Ana Con 1.

2.16 Use Fitch to give a formal version of the informal proof you gave in Exercise 2.5. Remember, you will find the problem setup in the file Exercise 2.16. You should begin your proof from this saved file. Save your completed proof as Proof 2.16.

In the following exercises, use Fitch to construct a formal proof that the conclusion is a consequence of the premises. Remember, begin your proof by opening the corresponding file, Exercise 2.x, *and save your solution as* Proof 2.x. *We're going to stop reminding you.*

2.17

$$\begin{array}{|l} \mathsf{SameCol}(a, b) \\ b = c \\ c = d \\ \hline \mathsf{SameCol}(a, d) \end{array}$$

2.18

$$\begin{array}{|l} \mathsf{Between}(a, d, b) \\ a = c \\ e = b \\ \hline \mathsf{Between}(c, d, e) \end{array}$$

2.19

$$\begin{array}{|l} \mathsf{Smaller}(a, b) \\ \mathsf{Smaller}(b, c) \\ \hline \mathsf{Smaller}(a, c) \end{array}$$

You will need to use **Ana Con** in this proof. This proof shows that the predicate Smaller in the blocks language is transitive.

2.20

$$\begin{array}{|l} \mathsf{RightOf}(b, c) \\ \mathsf{LeftOf}(d, e) \\ b = d \\ \hline \mathsf{LeftOf}(c, e) \end{array}$$

Make your proof parallel the informal proof we gave on page 52, using both an identity rule and **Ana Con** (where necessary).

3. Arrange the blocks so that the conclusion is false. Check the premises. If any of them are false, rearrange the blocks until they are all true. Is the conclusion still false? If not, keep trying. ◄

4. If you have trouble, try putting them in the order d, a, b, c. Now you will find that all the premises are true but the conclusion is false. This world is a counterexample to the argument. Thus we have demonstrated that the conclusion does not follow from the premises. ◄

5. Save your counterexample as World Counterexample 1. ◄

. . . . *Congratulations*

Remember

To demonstrate the invalidity of an argument with premises $\mathsf{P}_1, \ldots, \mathsf{P}_n$ and conclusion Q, find a counterexample: a possible circumstance that makes $\mathsf{P}_1, \ldots, \mathsf{P}_n$ all true but Q false. Such a counterexample shows that Q is not a consequence of $\mathsf{P}_1, \ldots, \mathsf{P}_n$.

Exercises

2.21 If you have skipped the **You try it** section, go back and do it now. Submit the world file World Counterexample 1.

2.22 ✎ Is the following argument valid? Sound? If it is valid, give an informal proof of it. If it is not valid, give an informal counterexample to it.

> All computer scientists are rich. Anyone who knows how to program a computer is a computer scientist. Bill Gates is rich. Therefore, Bill Gates knows how to program a computer.

2.23 ✎ Is the following argument valid? Sound? If it is valid, give an informal proof of it. If it is not valid, give an informal counterexample to it.

> Philosophers have the intelligence needed to be computer scientists. Anyone who becomes a computer scientist will eventually become wealthy. Anyone with the intelligence needed to be a computer scientist will become one. Therefore, every philosopher will become wealthy.

Each of the following problems presents a formal argument in the blocks language. If the argument is valid, submit a proof of it using Fitch. (You will find Exercise files for each of these in the usual place.) Important: if you use **Ana Con** *in your proof, cite at most two sentences in each application. If the argument is not valid, submit a counterexample world using Tarski's World.*

2.24

$$\begin{array}{|l} \mathsf{Larger(b, c)} \\ \mathsf{Smaller(b, d)} \\ \mathsf{SameSize(d, e)} \\ \hline \mathsf{Larger(e, c)} \end{array}$$

2.25

$$\begin{array}{|l} \mathsf{FrontOf(a, b)} \\ \mathsf{LeftOf(a, c)} \\ \mathsf{SameCol(a, b)} \\ \hline \mathsf{FrontOf(c, b)} \end{array}$$

2.26

$$\begin{array}{|l} \mathsf{SameRow(b, c)} \\ \mathsf{SameRow(a, d)} \\ \mathsf{SameRow(d, f)} \\ \mathsf{LeftOf(a, b)} \\ \hline \mathsf{LeftOf(f, c)} \end{array}$$

2.27

$$\begin{array}{|l} \mathsf{SameRow(b, c)} \\ \mathsf{SameRow(a, d)} \\ \mathsf{SameRow(d, f)} \\ \mathsf{FrontOf(a, b)} \\ \hline \mathsf{FrontOf(f, c)} \end{array}$$

Section 2.6

Alternative notation

You will often see arguments presented in the following way, rather than in Fitch format. The symbol ∴ (read "therefore") is used to indicate the conclusion:

All men are mortal.
Socrates is a man.
∴ Socrates is mortal.

There is a huge variety of formal deductive systems, each with its own notation. We can't possibly cover all of these alternatives, though we describe one, the resolution method, in Chapter 17.

Exercises

3.1 If you skipped the **You try it** section, go back and do it now. There are no files to submit, but you wouldn't want to miss it.

3.2 (Assessing negated sentences) Open Boole's World and Brouwer's Sentences. In the sentence file you will find a list of sentences built up from atomic sentences using only the negation symbol. Read each sentence and decide whether you think it is true or false. Check your assessment. If the sentence is false, make it true by adding or deleting a negation sign. When you have made all the sentences in the file true, submit the modified file as Sentences 3.2

3.3 (Building a world) Start a new sentence file. Write the following sentences in your file and save the file as Sentences 3.3.

1. $\neg$Tet(f)
2. $\neg$SameCol(c, a)
3. $\neg\neg$SameCol(c, b)
4. $\neg$Dodec(f)
5. c $\neq$ b
6. $\neg$(d $\neq$ e)
7. $\neg$SameShape(f, c)
8. $\neg\neg$SameShape(d, c)
9. $\neg$Cube(e)
10. $\neg$Tet(c)

Now start a new world file and build a world where all these sentences are true. As you modify the world to make the later sentences true, make sure that you have not accidentally falsified any of the earlier sentences. When you are done, submit both your sentences and your world.

3.4 Let P be a true sentence, and let Q be formed by putting some number of negation symbols in front of P. Show that if you put an even number of negation symbols, then Q is true, but that if you put an odd number, then Q is false. [Hint: A complete proof of this simple fact would require what is known as "mathematical induction." If you are familiar with proof by induction, then go ahead and give a proof. If you are not, just explain as clearly as you can why this is true.]

Now assume that P is atomic but of unknown truth value, and that Q is formed as before. No matter how many negation symbols Q has, it will always have the same truth value as a literal, namely either the literal P or the literal $\neg$P. Describe a simple procedure for determining which.

4. Play until Tarski's World says that you have lost. Then click on **Back** a couple of times, until you are back to where you are asked to choose a false conjunct. This time pick the false conjunct and resume the play of the game from that point. This time you will win. ◀

5. Notice that you can lose the game even when your original assessment is correct, if you make a bad choice along the way. But Tarski's World always allows you to back up and make different choices. If your original assessment is correct, there will always be a way to win the game. If it is impossible for you to win the game, then your original assessment was wrong. ◀

6. Save your sentence file as Sentences Game 1 when you are done. ◀

. . . . *Congratulations*

Remember

1. If P and Q are sentences of FOL, then so is $P \land Q$.
2. The sentence $P \land Q$ is true if and only if both P and Q are true.

Exercises

3.5 If you skipped the **You try it** section, go back and do it now. Make sure you follow all the instructions. Submit the file Sentences Game 1.

3.6 Start a new sentence file and open Wittgenstein's World. Write the following sentences in the sentence file.

1. Tet(f) ∧ Small(f)
2. Tet(f) ∧ Large(f)
3. Tet(f) ∧ ¬Small(f)
4. Tet(f) ∧ ¬Large(f)
5. ¬Tet(f) ∧ ¬Small(f)
6. ¬Tet(f) ∧ ¬Large(f)
7. ¬(Tet(f) ∧ Small(f))
8. ¬(Tet(f) ∧ Large(f))

9. ¬(¬Tet(f) ∧ ¬Small(f))
10. ¬(¬Tet(f) ∧ ¬Large(f))

Once you have written these sentences, decide which you think are true. Record your evaluations, to help you remember. Then go through and use Tarski's World to evaluate your assessments. Whenever you are wrong, play the game to see where you went wrong.

If you are never wrong, playing the game will not be very instructive. Play the game a couple times anyway, just for fun. In particular, try playing the game committed to the falsity of sentence 9. Since this sentence is true in Wittgenstein's World, Tarski's World should be able to beat you. Make sure you understand everything that happens as the game proceeds.

Next, change the size or shape of block f, predict how this will affect the truth values of your ten sentences, and see if your prediction is right. What is the maximum number of these sentences that you can get to be true in a single world? Build a world in which the maximum number of sentences are true. Submit both your sentence file and your world file, naming them as usual.

3.7 (Building a world) Open Max's Sentences. Build a world where all these sentences are true. You should start with a world with six blocks and make changes to it, trying to make all the sentences true. Be sure that as you make a later sentence true you do not inadvertently falsify an earlier sentence.

Section 3.3

Disjunction symbol: ∨

The symbol ∨ is used to express disjunction in our language, the notion we express in English using *or*. In first-order logic, this connective, like the conjunction sign, is always placed between two sentences, whereas in English we can also disjoin nouns, verbs, and other parts of speech. For example, the English sentences *John or Mary is home* and *John is home or Mary is home* both have the same first-order translation:

Home(john) ∨ Home(mary)

This FOL sentence is read "Home John or home Mary."

exclusive vs. inclusive disjunction

Although the English *or* is sometimes used in an "exclusive" sense, to say that *exactly* one (i.e., one but no more than one) of the two disjoined sentences is true, the first-order logic ∨ is always given an "inclusive" interpretation: it means that at least one and possibly both of the two disjoined sentences is true. Thus, our sample sentence is true if John is home but Mary is not, if Mary is home but John is not, or if both John and Mary are home.

of each, so Tarski's World will choose one and hold you to the commitment that it is false. (Tarski's World will, of course, try to win by picking a true one, if it can.)

You try it

► 1. Open the file Ackermann's World. Start a new sentence file and enter the sentence

$$\mathsf{Cube(c)} \lor \neg(\mathsf{Cube(a)} \lor \mathsf{Cube(b)})$$

Make sure you get the parentheses right!

► 2. Play the game committed (mistakenly) to this sentence being true. Since the sentence is a disjunction, and you are committed to TRUE, you will be asked to pick a disjunct that you think is true. Since the first one is obviously false, pick the second.

► 3. You now find yourself committed to the falsity of a (true) disjunction. Hence you are committed to the falsity of each disjunct. Tarski's World will then point out that you are committed to the falsity of Cube(b). But this is clearly wrong, since *b* is a cube. Continue until Tarski's World says you have lost.

► 4. Play the game again, this time committed to the falsity of the sentence. You should be able to win the game this time. If you don't, back up and try again.

► 5. Save your sentence file as Sentences Game 2

Congratulations

Remember

1. If P and Q are sentences of FOL, then so is P ∨ Q.
2. The sentence P ∨ Q is true if and only if P is true or Q is true (or both are true).

Exercises

3.8 If you skipped the **You try it** section, go back and do it now. You'll be glad you did. Well, maybe. Submit the file Sentences Game 2.

3.9 Open Wittgenstein's World and the sentence file Sentences 3.6 that you created for Exercise 3.6. Edit the sentences by replacing $\land$ by $\lor$ throughout, saving the edited list as Sentences 3.9. Once you have changed these sentences, decide which you think are true. Again, record your evaluations to help you remember them. Then go through and use Tarski's World to evaluate your assessment. Whenever you are wrong, play the game to see where you went wrong. If you are never wrong, then play the game anyway a couple times, knowing that you should win. As in Exercise 3.6, find the maximum number of sentences you can make true by changing the size or shape (or both) of block f. Submit both your sentences and world.

3.10 Open Ramsey's World and start a new sentence file. Type the following four sentences into the file:

1. $\mathsf{Between}(a, b, c) \lor \mathsf{Between}(b, a, c)$
2. $\mathsf{FrontOf}(a, b) \lor \mathsf{FrontOf}(c, b)$
3. $\neg\mathsf{SameRow}(b, c) \lor \mathsf{LeftOf}(b, a)$
4. $\mathsf{RightOf}(b, a) \lor \mathsf{Tet}(a)$

Assess each of these sentences in Ramsey's World and check your assessment. Then make a single change to the world that makes all four of the sentences come out false. Save the modified world as World 3.10. Submit both files.

Section 3.4

Remarks about the game

commitment and rules

We summarize the game rules for the three connectives, $\neg$, $\land$, and $\lor$, in Table 3.1. The first column indicates the form of the sentence in question, and the second indicates your current commitment, TRUE or FALSE. Which player moves depends on this commitment, as shown in the third column. The goal of that player's move is indicated in the final column. Notice that although the player to move depends on the commitment, the goal of that move does not depend on the commitment. You can see why this is so by thinking about the first row of the table, the one for $\mathsf{P} \lor \mathsf{Q}$. When you are committed to TRUE, it is clear that your goal should be to choose a true disjunct. But when you are committed to FALSE, Tarski's World is committed to TRUE, and so also has the same goal of choosing a true disjunct.

There is one somewhat subtle point that should be made about our way of describing the game. We have said, for example, that when you are committed to the truth of the disjunction $\mathsf{P} \lor \mathsf{Q}$, you are committed to the truth of one of the disjuncts. This of course is true, but does not mean you necessarily know which of P or Q is true. For example, if you have a sentence of the form

Exercises

Here is a problem that illustrates the remarks we made about sometimes being able to tell that a sentence is true, without knowing how to win the game.

3.11 ✎ Make sure Tarski's World is set to display the world in 3D. Then open Kleene's World and Kleene's Sentences. Some objects are hidden behind other objects, thus making it impossible to assess the truth of some of the sentences. Each of the six names a, b, c, d, e, and f are in use, naming some object. Now even though you cannot see all the objects, some of the sentences in the list can be evaluated with just the information at hand. Assess the truth of each claim, if you can, without recourse to the 2-D view. Then play the game. If your initial commitment is right, but you lose the game, back up and play over again. Then go through and add comments to each sentence explaining whether you can assess its truth in the world as shown, and why. Finally, display the 2-D view and check your work. We have annotated the first sentence for you to give you the idea. (The semicolon ";" tells Tarski's World that what follows is a comment.) When you are done, print out your annotated sentences to turn in to your instructor.

Section 3.5

Ambiguity and parentheses

When we first described FOL, we stressed the lack of ambiguity of this language as opposed to ordinary languages. For example, English allows us to say things like *Max is home or Claire is home and Carl is happy.* This sentence can be understood in two quite different ways. One reading claims that either Claire is home and Carl is happy, or Max is home. On this reading, the sentence would be true if Max was home, even if Carl was unhappy. The other reading claims both that Max or Claire is home and that Carl is happy.

FOL avoids this sort of ambiguity by requiring the use of parentheses, much the way they are used in algebra. So, for example, FOL would not have one sentence corresponding to the ambiguous English sentence, but two:

$$\mathsf{Home}(\mathsf{max}) \lor (\mathsf{Home}(\mathsf{claire}) \land \mathsf{Happy}(\mathsf{carl}))$$
$$(\mathsf{Home}(\mathsf{max}) \lor \mathsf{Home}(\mathsf{claire})) \land \mathsf{Happy}(\mathsf{carl})$$

The parentheses in the first indicate that it is a disjunction, whose second disjunct is itself a conjunction. In the second, they indicate that the sentence is a conjunction whose first conjunct is a disjunction. As a result, the truth conditions for the two are quite different. This is analogous to the difference in algebra between the expressions $2 + (x \times 3)$ and $(2 + x) \times 3$. This analogy between logic and algebra is one we will come back to later.

2. Evaluate each sentence in the file and check your assessment. If your assessment is wrong, play the game to see why. Don't go from one sentence to the next until you understand why it has the truth value it does. ◄

3. Do you see the importance of parentheses? After you understand all the sentences, go back and see which of the false sentences you can make true just by adding, deleting, or moving parentheses, but without making any other changes. Save your file as Sentences Ambiguity 1. ◄

. . . . ***Congratulations***

Exercises

To really master a new language, you have to use it, not just read about it. The exercises and problems that follow are intended to let you do just that.

3.12 If you skipped the **You try it** section, go back and do it now. Submit the file Sentences Ambiguity 1.

3.13 (Building a world) Open Schröder's Sentences. Build a single world where all the sentences in this file are true. As you work through the sentences, you will find yourself successively modifying the world. Whenever you make a change in the world, be careful that you don't make one of your earlier sentences false. When you are finished, verify that all the sentences are really true. Submit your world as World 3.13.

3.14 (Parentheses) Show that the sentence

$$\neg(\mathsf{Small}(\mathsf{a}) \lor \mathsf{Small}(\mathsf{b}))$$

is not a consequence of the sentence

$$\neg\mathsf{Small}(\mathsf{a}) \lor \mathsf{Small}(\mathsf{b})$$

You will do this by submitting a counterexample world in which the second sentence is true but the first sentence is false.

3.15 (More parentheses) Show that

$$\mathsf{Cube}(\mathsf{a}) \land (\mathsf{Cube}(\mathsf{b}) \lor \mathsf{Cube}(\mathsf{c}))$$

is not a consequence of the sentence

$$(\mathsf{Cube}(\mathsf{a}) \land \mathsf{Cube}(\mathsf{b})) \lor \mathsf{Cube}(\mathsf{c})$$

You will do this by submitting a counterexample world in which the second sentence is true but the first sentence is false.

3.16 (DeMorgan Equivalences) Open the file DeMorgan's Sentences. Construct a world where all the *odd* numbered sentences are true. Notice that no matter how you do this, the even numbered sentences also come out true. Submit this as World 3.16.1. Next build a world where all the odd numbered sentences are *false*. Notice that no matter how you do it, the even numbered sentences also come out false. Submit this as World 3.16.2.

3.17 ✎ In Exercise 3.16, you noticed an important fact about the relation between the even and odd numbered sentences in DeMorgan's Sentences. Try to explain why each even numbered sentence always has the same truth value as the odd numbered sentence that precedes it.

Section 3.6

Equivalent ways of saying things

Every language has many ways of saying the same thing. This is particularly true of English, which has absorbed a remarkable number of words from other languages in the course of its history. But in any language, speakers always have a choice of many synonymous ways of getting across their point. The world would be a boring place if there were just one way to make a given claim.

Fol is no exception, even though it is far less rich in its expressive capacities than English. In the blocks language, for example, none of our predicates is synonymous with another predicate, though it is obvious that we could do without many of them without cutting down on the claims expressible in the language. For instance, we could get by without the predicate RightOf by expressing everything we need to say in terms of the predicate LeftOf, systematically reversing the order of the names to get equivalent claims. This is not to say that RightOf means the same thing as LeftOf—it obviously does not—but just that the blocks language offers us a simple way to construct equivalent claims using these predicates. In the exercises at the end of this section, we explore a number of equivalences made possible by the predicates of the blocks language.

Some versions of fol are more parsimonious with their basic predicates than the blocks language, and so may not provide equivalent ways of expressing atomic claims. But even these languages cannot avoid multiple ways of expressing more complex claims. For example, $\mathsf{P} \land \mathsf{Q}$ and $\mathsf{Q} \land \mathsf{P}$ express the same claim in any first-order language. More interesting, because of the superficial differences in form, are the equivalences illustrated in Exercise 3.16, known as *DeMorgan's laws*. The first of DeMorgan's laws tells us that the negation of a conjunction, $\neg(\mathsf{P} \land \mathsf{Q})$, is logically equivalent to the disjunction of the negations of the original conjuncts: $\neg\mathsf{P} \lor \neg\mathsf{Q}$. The other tells us that the negation of a disjunction, $\neg(\mathsf{P} \lor \mathsf{Q})$, is equivalent to the conjunction of the negations of the original disjuncts: $\neg\mathsf{P} \land \neg\mathsf{Q}$. These laws are simple consequences of the meanings of the Boolean connectives. Writing $\mathsf{S}_1 \Leftrightarrow \mathsf{S}_2$ to indicate that S_1 and S_2 are logically equivalent, we can express DeMorgan's

DeMorgan's laws

laws in the following way:

$$\neg(\mathsf{P} \land \mathsf{Q}) \Leftrightarrow (\neg\mathsf{P} \lor \neg\mathsf{Q})$$
$$\neg(\mathsf{P} \lor \mathsf{Q}) \Leftrightarrow (\neg\mathsf{P} \land \neg\mathsf{Q})$$

There are many other equivalences that arise from the meanings of the Boolean connectives. Perhaps the simplest is known as the principle of *double negation*. Double negation says that a sentence of the form $\neg\neg\mathsf{P}$ is equivalent to the sentence P. We will systematically discuss these and other equivalences in the next chapter. In the meantime, we simply note these important equivalences before going on. Recognizing that there is more than one way of expressing a claim is essential before we tackle complicated claims involving the Boolean connectives.

double negation

Remember

(Double negation and DeMorgan's Laws) For any sentences P and Q:

1. Double negation: $\neg\neg\mathsf{P} \Leftrightarrow \mathsf{P}$
2. DeMorgan: $\neg(\mathsf{P} \land \mathsf{Q}) \Leftrightarrow (\neg\mathsf{P} \lor \neg\mathsf{Q})$
3. DeMorgan: $\neg(\mathsf{P} \lor \mathsf{Q}) \Leftrightarrow (\neg\mathsf{P} \land \neg\mathsf{Q})$

Exercises

3.18 (Equivalences in the blocks language) In the blocks language used in Tarski's World there are a number of equivalent ways of expressing some of the predicates. Open Bernays' Sentences. You will find a list of atomic sentences, where every other sentence is left blank. In each blank, write a sentence that is equivalent to the sentence above it, but does not use the predicate used in that sentence. (In doing this, you may presuppose any general facts about Tarski's World, for example that blocks come in only three shapes.) If your answers are correct, the odd numbered sentences will have the same truth values as the even numbered sentences in every world. Check that they do in Ackermann's World, Bolzano's World, Boole's World, and Leibniz's World. Submit the modified sentence file as Sentences 3.18.

3.19 ✎ (Equivalences in English) There are also equivalent ways of expressing predicates in English. For each of the following sentences of FOL, find an atomic sentence in English that expresses the same thing. For example, the sentence $\mathsf{Man(max)} \land \neg\mathsf{Married(max)}$ could be expressed in *is happy* is unambiguous, whereas it would be ambiguous without the *either*. What it means is that

$$[\mathsf{Home}(\mathsf{max}) \land \mathsf{Home}(\mathsf{claire})] \lor \mathsf{Happy}(\mathsf{carl})$$

In other words, *either* and *both* can sometimes act as left parentheses act in FOL. The same list of sentences demonstrates many other uses of *either* and *both*.

Remember

1. The English expression *and* sometimes suggests a temporal ordering; the FOL expression $\land$ never does.
2. The English expressions *but*, *however*, *yet*, *nonetheless*, and *moreover* are all stylistic variants of *and*.
3. The English expressions *either* and *both* are often used like parentheses to clarify an otherwise ambiguous sentence.

Exercises

3.20 (Describing a simple world) Open Boole's World. Start a new sentence file, named Sentences 3.20, where you will describe some features of this world. Check each of your sentences to see that it is indeed a sentence and that it is true in this world.

1. Notice that f (the large dodecahedron in the back) is not in front of a. Use your first sentence to say this.
2. Notice that f is to the right of a and to the left of b. Use your second sentence to say this.
3. Use your third sentence to say that f is either in back of or smaller than a.
4. Express the fact that both e and d are between c and a.
5. Note that neither e nor d is larger than c. Use your fifth sentence to say this.
6. Notice that e is neither larger than nor smaller than d. Use your sixth sentence to say this.
7. Notice that c is smaller than a but larger than e. State this fact.
8. Note that c is in front of f; moreover, it is smaller than f. Use your eighth sentence to state these things.

9. Notice that b is in the same row as a but is not in the same column as f. Use your ninth sentence to express this fact.
10. Notice that e is not in the same column as either c or d. Use your tenth sentence to state this.

Now let's change the world so that none of the above mentioned facts hold. We can do this as follows. First move f to the front right corner of the grid. (Be careful not to drop it off the edge. You might find it easier to make the move from the 2-D view. If you accidentally drop it, just open Boole's World again.) Then move e to the back left corner of the grid and make it large. Now none of the facts hold; if your answers to 1–10 are correct, all of the sentences should now be false. Verify that they are. If any are still true, can you figure out where you went wrong? Submit your sentences when you think they are correct. There is no need to submit the modified world file.

3.21 (Some translations) Tarski's World provides you with a very useful way to check whether your translation of a given English sentence is correct. If it is correct, then it will always have the same truth value as the English sentence, no matter what world the two are evaluated in. So when you are in doubt about one of your translations, simply build some worlds where the English sentence is true, others where it is false, and check to see that your translation has the right truth values in these worlds. You should use this technique frequently in all of the translation exercises.

Start a new sentence file, and use it to enter translations of the following English sentences into first-order logic. You will only need to use the connectives $\land, \lor$, and $\neg$.

1. *Either **a** is small or both **c** and **d** are large.*
2. ***d** and **e** are both in back of **b**.*
3. ***d** and **e** are both in back of **b** and larger than it.*
4. *Both **d** and **c** are cubes, however neither of them is small.*
5. *Neither **e** nor **a** is to the right of **c** and to the left of **b**.*
6. *Either **e** is not large or it is in back of **a**.*
7. ***c** is neither between **a** and **b**, nor in front of either of them.*
8. *Either both **a** and **e** are tetrahedra or both **a** and **f** are.*
9. *Neither **d** nor **c** is in front of either **c** or **b**.*
10. ***c** is either between **d** and **f** or smaller than both of them.*
11. *It is not the case that **b** is in the same row as **c**.*
12. ***b** is in the same column as **e**, which is in the same row as **d**, which in turn is in the same column as **a**.*

Before you submit your sentence file, do the next exercise.

3.22 (Checking your translations) Open Wittgenstein's World. Notice that all of the English sentences from Exercise 3.21 are true in this world. Thus, if your translations are accurate, they will also be true in this world. Check to see that they are. If you made any mistakes, go back and fix them. But as we have stressed, even if one of your sentences comes out true in Wittgenstein's World, it does not mean that it is a proper translation of the corresponding English sentence. All you know for sure is that your translation and the original sentence have the same truth value in this particular world. If the translation is correct, it will have the same truth value as the English sentence in *every* world. Thus, to have a better test of your translations, we will examine them in a number of worlds, to see if they have the same truth values as their English counterparts in all of these worlds.

Let's start by making modifications to Wittgenstein's World. Make all the large or medium objects small, and the small objects large. With these changes in the world, the English sentences 1, 3, 4, and 10 become false, while the rest remain true. Verify that the same holds for your translations. If not, correct your translations. Next, rotate your modified Wittgenstein's World 90° clockwise. Now sentences 5, 6, 8, 9, and 11 should be the only true ones that remain.

Let's check your translations in another world. Open Boole's World. The only English sentences that are true in this world are sentences 6 and 11. Verify that all of your translations except 6 and 11 are false. If not, correct your translations.

Now modify Boole's World by exchanging the positions of *b* and *c*. With this change, the English sentences 2, 5, 6, 7, and 11 come out true, while the rest are false. Check that the same is true of your translations.

There is nothing to submit except Sentences 3.21.

3.23 Start a new sentence file and translate the following into FOL. Use the names and predicates presented in Table 1.2 on page 30.

1. *Max is a student, not a pet.*
2. *Claire fed Folly at 2 pm and then ten minutes later gave her to Max.*
3. *Folly belonged to either Max or Claire at 2:05 pm.*
4. *Neither Max nor Claire fed Folly at 2 pm or at 2:05 pm.*
5. *2:00 pm is between 1:55 pm and 2:05 pm.*
6. *When Max gave Folly to Claire at 2 pm, Folly wasn't hungry, but she was an hour later.*

3.24 Referring again to Table 1.2, page 30, translate the following into natural, colloquial English. Turn in your translations to your instructor.

1. Student(claire) ∧ ¬Student(max)
2. Pet(pris) ∧ ¬Owned(max, pris, 2:00)
3. Owned(claire, pris, 2:00) ∨ Owned(claire, folly, 2:00)
4. ¬(Fed(max, pris, 2:00) ∧ Fed(max, folly, 2:00))

5. $((\mathsf{Gave}(\mathsf{max}, \mathsf{pris}, \mathsf{claire}, 2{:}00) \wedge \mathsf{Hungry}(\mathsf{pris}, 2{:}00)) \vee$
   $(\mathsf{Gave}(\mathsf{max}, \mathsf{folly}, \mathsf{claire}, 2{:}00) \wedge \mathsf{Hungry}(\mathsf{folly}, 2{:}00))) \wedge$
   $\mathsf{Angry}(\mathsf{claire}, 2{:}05)$

3.25 ✎⋆ Translate the following into FOL, introducing names, predicates, and function symbols as needed. Explain the meaning of each predicate and function symbol, unless it is completely obvious.

1. *AIDS is less contagious than influenza, but more deadly.*
2. *Abe fooled Stephen on Sunday, but not on Monday.*
3. *Sean or Brad admires Meryl and Harrison.*
4. *Daisy is a jolly miller, and lives on the River Dee.*
5. *Polonius's eldest child was neither a borrower nor a lender.*

3.26 ✎ (Boolean solids) Many of you know how to do a "Boolean search" on the Web or on your computer. When we do a Boolean search, we are really using a generalization of the Boolean truth functions. We specify a Boolean combination of words as a criterion for finding documents that contain (or do not contain) those words. Another generalization of the Boolean operations is to spatial objects. In Figure 3.1 we show four ways to combine a vertical cylinder (A) with a horizontal cylinder (B) to yield a new solid. Give an intuitive explanation of how the Boolean connectives are being applied in this example. Then describe what the object $\neg(\mathsf{A} \wedge \mathsf{B})$ would be like and explain why we didn't give you a picture of this solid.

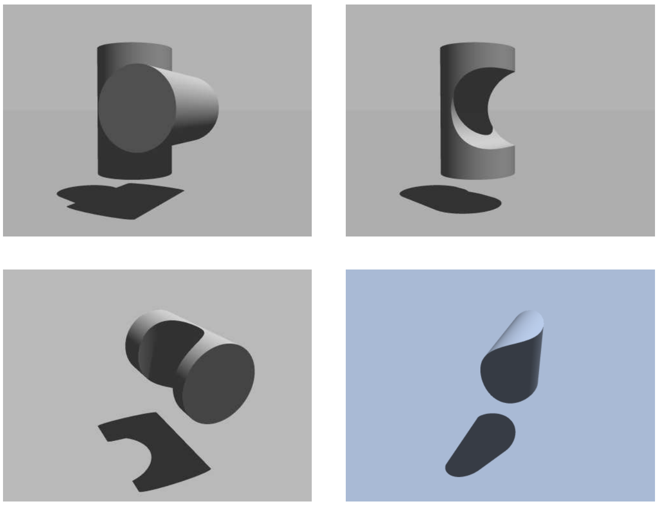

Figure 3.1: Boolean combinations of solids: $\mathsf{A} \vee \mathsf{B}$, $\mathsf{A} \wedge \neg\mathsf{B}$, $\neg\mathsf{A} \wedge \mathsf{B}$, and $\mathsf{A} \wedge \mathsf{B}$.

Exercises

3.27 (Overcoming dialect differences) The following are all sentences of FOL. But they're in different dialects. Submit a sentence file in which you've translated them into our dialect.

1. $\overline{\overline{\mathsf{P}}\&\overline{\mathsf{Q}}}$
2. $!(\mathsf{P} \parallel (\mathsf{Q}\&\&\mathsf{P}))$
3. $(\sim \mathsf{P} \vee \mathsf{Q}) \cdot \mathsf{P}$
4. $\mathsf{P}(\sim \mathsf{Q} \vee \mathsf{RS})$

3.28 (Translating from Polish) Try your hand at translating the following sentences from Polish notation into our dialect. Submit the resulting sentence file.

1. NKpq
2. KNpq
3. NAKpqArs
4. NAKpAqrs
5. NAKApqrs

Exercises

In this chapter, you will often be using Boole to construct truth tables. Although Boole has the capability of building and filling in reference columns for you, do not use this feature. To understand truth tables, you need to be able to do this yourself. In later chapters, we will let you use the feature, once you've learned how to do it yourself. The Grade Grinder will, by the way, be able to tell if Boole constructed the reference columns.

4.1 If you skipped the **You try it** section, go back and do it now. Submit the file Table Tautology 1.

4.2 Assume that A, B, and C are atomic sentences. Use Boole to construct truth tables for each of the following sentences and, based on your truth tables, say which are tautologies. Name your tables Table 4.2.x, where x is the number of the sentence.

1. $(A \land B) \lor (\neg A \lor \neg B)$
2. $(A \land B) \lor (A \land \neg B)$
3. $\neg(A \land B) \lor C$
4. $(A \lor B) \lor \neg(A \lor (B \land C))$

4.3 ★ In Exercise 4.2 you should have discovered that two of the four sentences are tautologies, and hence logical truths.

1. Suppose you are told that the atomic sentence A is in fact a logical truth (for example, $a = a$). Can you determine whether any additional sentences in the list (1)-(4) are logically necessary based on this information?
2. Suppose you are told that A is in fact a logically false sentence (for example, $a \neq a$). Can you determine whether any additional sentences in the list (1)-(4) are logical truths based on this information?

In the following four exercises, use Boole to construct truth tables and indicate whether the sentence is TT*-possible and whether it is a tautology. Remember how you should treat long conjunctions and disjunctions.*

4.4 $\neg(B \land \neg C \land \neg B)$

4.5 $A \lor \neg(B \lor \neg(C \land A))$

4.6 $\neg[\neg A \lor \neg(B \land C) \lor (A \land B)]$

4.7 $\neg[(\neg A \lor B) \land \neg(C \land D)]$

4.8 Make a copy of the Euler circle diagram on page 102 and place the numbers of the following sentences in the appropriate region.

1. $a = b$
2. $a = b \lor b = b$

3. $\mathsf{a = b \land b = b}$
4. $\neg(\mathsf{Large(a)} \land \mathsf{Large(b)} \land \mathsf{Adjoins(a, b)})$
5. $\mathsf{Larger(a, b)} \lor \neg\mathsf{Larger(a, b)}$
6. $\mathsf{Larger(a, b)} \lor \mathsf{Smaller(a, b)}$
7. $\neg\mathsf{Tet(a)} \lor \neg\mathsf{Cube(b)} \lor \mathsf{a \neq b}$
8. $\neg(\mathsf{Small(a)} \land \mathsf{Small(b)}) \lor \mathsf{Small(a)}$
9. $\mathsf{SameSize(a, b)} \lor \neg(\mathsf{Small(a)} \land \mathsf{Small(b)})$
10. $\neg(\mathsf{SameCol(a, b)} \land \mathsf{SameRow(a, b)})$

4.9 ➶|✎ (Logical dependencies) Use Tarski's World to open Weiner's Sentences.

1. For each of the ten sentences in this file, construct a truth table in Boole and assess whether the sentence is TT-possible. Name your tables Table 4.9.x, where x is the number of the sentence in question. Use the results to fill in the first column of the following table:

Sentence	TT-possible	TW-possible
1		
2		
3		
⋮		
10		

2. In the second column of the table, put *yes* if you think the sentence is TW-possible, that is, if it is possible to make the sentence true by building a world in Tarski's World, and *no* otherwise. For each sentence that you mark TW-possible, actually build a world in which it is true and name it World 4.9.x, where x is the number of the sentence in question. The truth tables you constructed before may help you build these worlds.
3. Are any of the sentences TT-possible but not TW-possible? Explain why this can happen. Are any of the sentences TW-possible but not TT-possible? Explain why not.

Submit the files you created and turn in the table and explanations to your instructor.

4.10 ✎⋆ Draw an Euler circle diagram similar to the diagram on page 102, but this time showing the relationship between the notions of logical possibility, TW-possibility, and TT-possibility. For each region in the diagram, indicate an example sentence that would fall in that region. Don't forget the region that falls outside all the circles.

All necessary truths are obviously possible: since they are true in *all* possible circumstances, they are surely true in *some* possible circumstances. Given this reflection, where would the sentences from our previous diagram on page 102 fit into the new diagram?

4.11 ✎⋆⋆ Suppose that S is a tautology, with atomic sentences A, B, and C. Suppose that we replace all occurrences of A by another sentence P, possibly complex. Explain why the resulting sentence is still a tautology. This is expressed by saying that substitution preserves tautologicality. Explain why substitution of atomic sentences does not always preserve logical truth, even though it preserves tautologies. Give an example.

Section 4.2

Logical and tautological equivalence

In the last chapter, we introduced the notion of logically equivalent sentences, sentences that have the same truth values in every possible circumstance. When two sentences are logically equivalent, we also say they have the same truth conditions, since the conditions under which they come out true or false are identical.

logical equivalence

The notion of logical equivalence, like logical necessity, is somewhat vague, but not in a way that prevents us from studying it with precision. For here too we can introduce precise concepts that bear a clear relationship to the intuitive notion we aim to understand better. The key concept we will introduce in this section is that of *tautological equivalence.* Two sentences are tautologically equivalent if they can be seen to be equivalent simply in virtue of the meanings of the truth-functional connectives. As you might expect, we can check for tautological equivalence using truth tables.

tautological equivalence

Suppose we have two sentences, S and S′, that we want to check for tautological equivalence. What we do is construct a truth table with a reference column for each of the atomic sentences that appear in either of the two sentences. To the right, we write both S and S′, with a vertical line separating them, and fill in the truth values under the connectives as usual. We call this a *joint* truth table for the sentences S and S′. When the joint truth table is completed, we compare the column under the main connective of S with the column under the main connective of S′. If these columns are identical, then we know that the truth conditions of the two sentences are the same.

joint truth tables

Let's look at an example. Using A and B to stand for arbitrary atomic sentences, let us test the first DeMorgan law for tautological equivalence. We would do this by means of the following joint truth table.

A	B	$\neg$(A	$\wedge$ B)	$\neg$A	$\vee$	$\neg$B
T	T	**F**	T	F	**F**	F
T	F	**T**	F	F	**T**	T
F	T	**T**	F	T	**T**	F
F	F	**T**	F	T	**T**	T

In this table, the columns in bold correspond to the main connectives of the

Remember

Let S and S′ be a sentences of FOL built up from atomic sentences by means of truth-functional connectives alone. To test for tautological equivalence, we construct a *joint* truth table for the two sentences.

1. S and S′ are *tautologically equivalent* if and only if every row of the joint truth table assigns the same values to S and S′.
2. If S and S′ are tautologically equivalent, then they are logically equivalent.
3. Some logically equivalent sentences are not tautologically equivalent.

Exercises

In Exercises 4.12-4.18, use Boole to construct joint truth tables showing that the pairs of sentences are logically (indeed, tautologically) equivalent. To add a second sentence to your joint truth table, choose **Add Column After** *from the* **Table** *menu. Don't forget to specify your assessments, and remember, you should build and fill in your own reference columns.*

4.12 (DeMorgan)
$\neg(A \lor B)$ and $\neg A \land \neg B$

4.13 (Associativity)
$(A \land B) \land C$ and $A \land (B \land C)$

4.14 (Associativity)
$(A \lor B) \lor C$ and $A \lor (B \lor C)$

4.15 (Idempotence)
$A \land B \land A$ and $A \land B$

4.16 (Idempotence)
$A \lor B \lor A$ and $A \lor B$

4.17 (Distribution)
$A \land (B \lor C)$ and $(A \land B) \lor (A \land C)$

4.18 (Distribution)
$A \lor (B \land C)$ and $(A \lor B) \land (A \lor C)$

4.19 (TW-equivalence) Suppose we introduced the notion of TW-equivalence, saying that two sentences of the blocks language are TW-equivalent if and only if they have the same truth value in every world that can be constructed in Tarski's World.

1. What is the relationship between TW-equivalence, tautological equivalence and logical equivalence?
2. Give an example of a pair of sentences that are TW-equivalent but not logically equivalent.

Remember

Let $\mathsf{P}_1, \ldots, \mathsf{P}_n$ and Q be sentences of FOL built up from atomic sentences by means of truth functional connectives alone. Construct a joint truth table for all of these sentences.

1. Q is a tautological consequence of $\mathsf{P}_1, \ldots, \mathsf{P}_n$ if and only if every row that assigns **T** to each of $\mathsf{P}_1, \ldots, \mathsf{P}_n$ also assigns **T** to Q.
2. If Q is a tautological consequence of $\mathsf{P}_1, \ldots, \mathsf{P}_n$, then Q is also a logical consequence of $\mathsf{P}_1, \ldots, \mathsf{P}_n$.
3. Some logical consequences are not tautological consequences.

Exercises

For each of the arguments below, use the truth table method to determine whether the conclusion is a tautological consequence of the premises. Your truth table for Exercise 4.24 will be fairly large. It's good for the soul to build a large truth table every once in a while. Be thankful you have Boole to help you. (But make sure you build your own reference columns!)

4.20

(Tet(a) ∧ Small(a)) ∨ Small(b)

Small(a) ∨ Small(b)

4.21

Taller(claire, max) ∨ Taller(max, claire)
Taller(claire, max)

¬Taller(max, claire)

4.22

Large(a)
Cube(a) ∨ Dodec(a)

(Cube(a) ∧ Large(a)) ∨ (Dodec(a) ∧ Large(a))

4.23 ★

A ∨ ¬B
B ∨ C
C ∨ D

A ∨ ¬D

4.24 ★

¬A ∨ B ∨ C
¬C ∨ D
¬(B ∧ ¬E)

D ∨ ¬A ∨ E

4.25 ★

Give an example of two different sentences A and B in the blocks language such that A ∧ B is a logical consequence of A ∨ B. [Hint: Note that A ∧ A is a logical consequence of A ∨ A, but here we insist that A and B be distinct sentences.]

You try it

1. Open the file Taut Con 2. You will find a proof containing ten steps whose rules have not been specified. ◀

2. Focus on each step in turn. You will find that the supporting steps have already been cited. Convince yourself that the step follows from the cited sentences. Is it a tautological consequence of the sentences cited? If so, change the rule to **Taut Con** and see if you were right. If not, change it to **Ana Con** and see if it checks out. (If **Taut Con** will work, make sure you use it rather than the stronger **Ana Con**.) ◀

3. When all of your steps check out using **Taut Con** or **Ana Con**, go back and find the one step whose rule can be changed from **Ana Con** to the weaker **FO Con**. ◀

4. When each step checks out using the weakest **Con** rule possible, save your proof as Proof Taut Con 2. ◀

Congratulations

Exercises

4.26 If you skipped the **You try it** sections, go back and do them now. Submit the files Proof Taut Con 1 and Proof Taut Con 2.

For each of the following arguments, decide whether the conclusion is a tautological consequence of the premises. If it is, submit a proof that establishes the conclusion using one or more applications of **Taut Con**. *Do not cite more than two sentences at a time for any of your applications of* **Taut Con**. *If the conclusion is not a consequence of the premises, submit a counterexample world showing that the argument is not valid.*

4.27

| Cube(a) ∨ Cube(b)
| Dodec(c) ∨ Dodec(d)
| ¬Cube(a) ∨ ¬Dodec(c)
|—
| Cube(b) ∨ Dodec(d)

4.28

| Large(a) ∨ Large(b)
| Large(a) ∨ Large(c)
|—
| Large(a) ∧ (Large(b) ∨ Large(c))

4.29

$$\begin{array}{|l} \mathsf{Small(a)} \lor \mathsf{Small(b)} \\ \mathsf{Small(b)} \lor \mathsf{Small(c)} \\ \mathsf{Small(c)} \lor \mathsf{Small(d)} \\ \mathsf{Small(d)} \lor \mathsf{Small(e)} \\ \neg\mathsf{Small(c)} \\ \hline \mathsf{Small(a)} \lor \mathsf{Small(e)} \end{array}$$

4.30

$$\begin{array}{|l} \mathsf{Tet(a)} \lor \neg(\mathsf{Tet(b)} \land \mathsf{Tet(c)}) \\ \neg(\neg\mathsf{Tet(b)} \lor \neg\mathsf{Tet(d)}) \\ (\mathsf{Tet(e)} \land \mathsf{Tet(c)}) \lor (\mathsf{Tet(c)} \land \mathsf{Tet(d)}) \\ \hline \mathsf{Tet(a)} \end{array}$$

Section 4.5

Pushing negation around

When two sentences are logically equivalent, each is a logical consequence of the other. As a result, in giving an informal proof, you can always go from an established sentence to one that is logically equivalent to it. This fact makes observations like the DeMorgan laws and double negation quite useful in giving informal proofs.

substitution of logical equivalents

What makes these equivalences even more useful is the fact that logically equivalent sentences can be substituted for one another in the context of a larger sentence and the resulting sentences will also be logically equivalent. An example will help illustrate what we mean. Suppose we start with the sentence:

$$\neg(\mathsf{Cube(a)} \land \neg\neg\mathsf{Small(a)})$$

By the principle of double negation, we know that $\mathsf{Small(a)}$ is logically equivalent to $\neg\neg\mathsf{Small(a)}$. Since these have exactly the same truth conditions, we can substitute $\mathsf{Small(a)}$ for $\neg\neg\mathsf{Small(a)}$ in the context of the above sentence, and the result,

$$\neg(\mathsf{Cube(a)} \land \mathsf{Small(a)})$$

will be logically equivalent to the original, a fact that you can check by constructing a joint truth table for the two sentences.

We can state this important fact in the following way. Let's write $\mathsf{S(P)}$ for an FOL sentence that contains the (possibly complex) sentence P as a component part, and $\mathsf{S(Q)}$ for the result of substituting Q for P in $\mathsf{S(P)}$. Then if P and Q are logically equivalent:

$$\mathsf{P} \Leftrightarrow \mathsf{Q}$$

it follows that $\mathsf{S(P)}$ and $\mathsf{S(Q)}$ are also logically equivalent:

We call a demonstration of this sort a *chain of equivalences*. The first step in this chain is justified by one of the DeMorgan laws. The second step involves two applications of double negation. In the next step we use associativity to remove the unnecessary parentheses. In the fourth step, we use idempotence of $\lor$. The next to the last step uses commutativity of $\lor$, while the final step uses idempotence of $\land$.

chain of equivalences

Remember

1. *Substitution of equivalents:* If P and Q are logically equivalent:

$$\mathsf{P} \Leftrightarrow \mathsf{Q}$$

then the results of substituting one for the other in the context of a larger sentence are also logically equivalent:

$$\mathsf{S(P)} \Leftrightarrow \mathsf{S(Q)}$$

2. A sentence is in *negation normal form* (NNF) if all occurrences of $\neg$ apply directly to atomic sentences.
3. Any sentence built from atomic sentences using just $\land$, $\lor$, and $\neg$ can be put into negation normal form by repeated application of the DeMorgan laws and double negation.
4. Sentences can often be further simplified using the principles of associativity, commutativity, and idempotence.

Exercises

4.31 (Negation normal form) Use Tarski's World to open Turing's Sentences. You will find the following five sentences, each followed by an empty sentence position.

1. $\neg(\mathsf{Cube(a)} \land \mathsf{Larger(a,b)})$
3. $\neg(\mathsf{Cube(a)} \lor \neg\mathsf{Larger(b,a)})$
5. $\neg(\neg\mathsf{Cube(a)} \lor \neg\mathsf{Larger(a,b)} \lor \mathsf{a} \neq \mathsf{b})$
7. $\neg(\mathsf{Tet(b)} \lor (\mathsf{Large(c)} \land \neg\mathsf{Smaller(d,e)}))$
9. $\mathsf{Dodec(f)} \lor \neg(\mathsf{Tet(b)} \lor \neg\mathsf{Tet(f)} \lor \neg\mathsf{Dodec(f)})$

In the empty positions, write the negation normal form of the sentence above it. Then build any world where all of the names are in use. If you have gotten the negation normal forms correct, each even numbered sentence will have the same truth value in your world as the odd numbered sentence above it. Verify that this is so in your world. Submit the modified sentence file as Sentences 4.31.

4.32 (Negation normal form) Use Tarski's World to open the file Sextus' Sentences. In the odd numbered slots, you will find the following sentences.

1. ¬(Home(carl) ∧ ¬Home(claire))
3. ¬[Happy(max) ∧ (¬Likes(carl, claire) ∨ ¬Likes(claire, carl))]
5. ¬¬¬[(Home(max) ∨ Home(carl)) ∧ (Happy(max) ∨ Happy(carl))]

Use Double Negation and DeMorgan's laws to put each sentence into negation normal form in the slot below it. Submit the modified file as Sentences 4.32.

In each of the following exercises, use associativity, commutativity, and idempotence to simplify the sentence as much as you can using just these rules. Your answer should consist of a chain of logical equivalences like the chain given on page 120. At each step of the chain, indicate which principle you are using.

4.33 (A ∧ B) ∧ A

4.34 (B ∧ (A ∧ B ∧ C))

4.35 (A ∨ B) ∨ (C ∧ D) ∨ A

4.36 (¬A ∨ B) ∨ (B ∨ C)

4.37 (A ∧ B) ∨ C ∨ (B ∧ A) ∨ A

Section 4.6

Conjunctive and disjunctive normal forms

We have seen that with a few simple principles of Boolean logic, we can start with a sentence and transform it into a logically equivalent sentence in negation normal form, one where all negations occur in front of atomic sentences. We can improve on this by introducing the so-called distributive laws. These additional equivalences will allow us to transform sentences into what are known as *conjunctive normal form* (CNF) and *disjunctive normal form* (DNF). These normal forms are quite important in certain applications of logic in computer science, as we discuss in Chapter 17. We will also use disjunctive normal form to demonstrate an important fact about the Boolean connectives in Chapter 7.

distribution

Recall that in algebra you learned that multiplication distributes over addition: $a \times (b+c) = (a \times b)+(a \times c)$. The distributive laws of logic look formally

► 4. In general it is easier to evaluate the truth value of a sentence in disjunctive normal form. This comes out in the game, which takes at most three steps for a sentence in DNF, one each for $\lor$, $\land$, and $\neg$, in that order. There is no limit to the number of steps a sentence in other forms may take.

► 5. Save the world you have created as World DNF 1.

. . . . ***Congratulations***

Exercises

4.38 If you skipped the **You try it** section, go back and do it now. Submit the file World DNF 1.

4.39 Open CNF Sentences. In this file you will find the following conjunctive normal form sentences in the odd numbered positions, but you will see that the even numbered positions are blank.

1. $(\mathsf{LeftOf}(a, b) \lor \mathsf{BackOf}(a, b)) \land \mathsf{Cube}(a)$
3. $\mathsf{Larger}(a, b) \land (\mathsf{Cube}(a) \lor \mathsf{Tet}(a) \lor a = b)$
5. $(\mathsf{Between}(a, b, c) \lor \mathsf{Tet}(a) \lor \neg\mathsf{Tet}(b)) \land \mathsf{Dodec}(c)$
7. $\mathsf{Cube}(a) \land \mathsf{Cube}(b) \land (\neg\mathsf{Small}(a) \lor \neg\mathsf{Small}(b))$
9. $(\mathsf{Small}(a) \lor \mathsf{Medium}(a)) \land (\mathsf{Cube}(a) \lor \neg\mathsf{Dodec}(a))$

In the even numbered positions you should fill in a DNF sentence logically equivalent to the sentence above it. Check your work by opening several worlds and checking to see that each of your sentences has the same truth value as the one above it. Submit the modified file as Sentences 4.39.

4.40 Open More CNF Sentences. In this file you will find the following sentences in every third position.

1. $\neg[(\mathsf{Cube}(a) \land \neg\mathsf{Small}(a)) \lor (\neg\mathsf{Cube}(a) \land \mathsf{Small}(a))]$
4. $\neg[(\mathsf{Cube}(a) \lor \neg\mathsf{Small}(a)) \land (\neg\mathsf{Cube}(a) \lor \mathsf{Small}(a))]$
7. $\neg(\mathsf{Cube}(a) \land \mathsf{Larger}(a, b)) \land \mathsf{Dodec}(b)$
10. $\neg(\neg\mathsf{Cube}(a) \land \mathsf{Tet}(b))$
13. $\neg\neg\mathsf{Cube}(a) \lor \mathsf{Tet}(b)$

The two blanks that follow each sentence are for you to first transform the sentence into negation normal form, and then put that sentence into CNF. Again, check your work by opening several worlds to see that each of your sentences has the same truth value as the original. When you are finished, submit the modified file as Sentences 4.40.

In Exercises 4.41-4.43, use a chain of equivalences to convert each sentence into an equivalent sentence in disjunctive normal form. Simplify your answer as much as possible using the laws of associativity, commutativity, and idempotence. At each step in your chain, indicate which principle you are applying. Assume that $\mathsf{A}, \mathsf{B}, \mathsf{C},$ *and* D *are literals.*

4.41 $\mathsf{C} \wedge (\mathsf{A} \vee (\mathsf{B} \wedge \mathsf{C}))$
✎

4.42 $\mathsf{B} \wedge (\mathsf{A} \wedge \mathsf{B} \wedge (\mathsf{A} \vee \mathsf{B} \vee (\mathsf{B} \wedge \mathsf{C})))$
✎

4.43 $\mathsf{A} \wedge (\mathsf{A} \wedge (\mathsf{B} \vee (\mathsf{A} \wedge \mathsf{C})))$
✎

Exercises

In the following exercises we list a number of patterns of inference, only some of which are valid. For each pattern, determine whether it is valid. If it is, explain why it is valid, appealing to the truth tables for the connectives involved. If it is not, give a specific example of how the step could be used to get from true premises to a false conclusion.

5.1 ✎ From $\mathsf{P} \lor \mathsf{Q}$ and $\neg\mathsf{P}$, infer Q.

5.2 ✎ From $\mathsf{P} \lor \mathsf{Q}$ and Q, infer $\neg\mathsf{P}$.

5.3 ✎ From $\neg(\mathsf{P} \lor \mathsf{Q})$, infer $\neg\mathsf{P}$.

5.4 ✎ From $\neg(\mathsf{P} \land \mathsf{Q})$ and P, infer $\neg\mathsf{Q}$.

5.5 ✎ From $\neg(\mathsf{P} \land \mathsf{Q})$, infer $\neg\mathsf{P}$.

5.6 ✎* From $\mathsf{P} \land \mathsf{Q}$ and $\neg\mathsf{P}$, infer R.

Section 5.2

Proof by cases

The simple forms of inference discussed in the last section are all instances of the principle that you can use already established cases of logical consequence in informal proofs. But the Boolean connectives also give rise to two entirely new methods of proof, methods that are explicitly applied in all types of rigorous reasoning. The first of these is the method of *proof by cases*. In our formal system $\mathcal{F}$, this method will be called *disjunction elimination*, but don't be misled by the ordinary sounding name: it is far more significant than, say, disjunction introduction or conjunction elimination.

We begin by illustrating proof by cases with a well-known piece of mathematical reasoning. The reasoning proves that there are irrational numbers b and c such that b^c is rational. First, let's review what this means. A number is said to be *rational* if it can be expressed as a fraction n/m, for integers n and m. If it can't be so expressed, then it is irrational. Thus 2 is rational $(2 = 2/1)$, but $\sqrt{2}$ is irrational. (We will prove this latter fact in the next section, to illustrate proof by contradiction; for now, just take it as a well-known truth.) Here now is our proof:

> **Proof:** To show that there are irrational numbers b and c such that b^c is rational, we will consider the number $\sqrt{2}^{\sqrt{2}}$. We note that this number is either rational or irrational.

5.7

Home(max) ∨ Home(claire)
¬Home(max) ∨ Happy(carl)
¬Home(claire) ∨ Happy(carl)

Happy(carl)

5.8

LeftOf(a, b) ∨ RightOf(a, b)
BackOf(a, b) ∨ ¬LeftOf(a, b)
FrontOf(b, a) ∨ ¬RightOf(a, b)
SameCol(c, a) ∧ SameRow(c, b)

BackOf(a, b)

5.9 Assume the same four premises as in Exercise 5.8. Is LeftOf(b, c) a logical consequence of these premises? If so, turn in an informal proof of the argument's validity. If not, submit a counterexample world.

5.10 Suppose Max's favorite basketball team is the Chicago Bulls and favorite football team is the Denver Broncos. Max's father John is returning from Indianapolis to San Francisco on United Airlines, and promises that he will buy Max a souvenir from one of his favorite teams on the way. Explain John's reasoning, appealing to the annoying fact that all United flights between Indianapolis and San Francisco stop in either Denver or Chicago. Make explicit the role proof by cases plays in this reasoning.

5.11 Suppose the police are investigating a burglary and discover the following facts. All the doors to the house were bolted from the inside and show no sign of forced entry. In fact, the only possible ways in and out of the house were a small bathroom window on the first floor that was left open and an unlocked bedroom window on the second floor. On the basis of this, the detectives rule out a well-known burglar, Julius, who weighs two hundred and fifty pounds and is arthritic. Explain their reasoning.

5.12 In our proof that there are irrational numbers b and c where b^c is rational, one of our steps was to assert that $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational. What justifies the introduction of this claim into our proof?

5.13 Describe an everyday example of reasoning by cases that you have performed in the last few days.

5.14★ Give an informal proof that if S is a tautological consequence of P and a tautological consequence of Q, then S is a tautological consequence of P ∨ Q. Remember that the joint truth table for P ∨ Q and S may have more rows than either the joint truth table for P and S, or the joint truth table for Q and S. [Hint: Assume you are looking at a single row of the joint truth table for P ∨ Q and S in which P ∨ Q is true. Break into cases based on whether P is true or Q is true and prove that S must be true in either case.]

Exercises

In the following exercises, decide whether the displayed argument is valid. If it is, turn in an informal proof, phrased in complete, well-formed English sentences, making use of first-order sentences as convenient. Whenever you use proof by cases or proof by contradiction, say so. You don't have to be explicit about the use of simple proof steps like conjunction elimination. If the argument is invalid, construct a counterexample world in Tarski's World. (Argument 5.16 is valid, and so will not require a counterexample.)

5.15 ➶|✎

| *b* is a tetrahedron.
| *c* is a cube.
| Either *c* is larger than *b* or else they are identical.
|—
| *b* is smaller than *c*.

5.16 ✎

| Max or Claire is at home but either Scruffy or Carl is unhappy.
| Either Max is not home or Carl is happy.
| Either Claire is not home or Scruffy is unhappy.
|—
| Scruffy is unhappy.

5.17 ➶|✎

| $\mathsf{Cube(a)} \lor \mathsf{Tet(a)} \lor \mathsf{Large(a)}$
| $\neg\mathsf{Cube(a)} \lor \mathsf{a} = \mathsf{b} \lor \mathsf{Large(a)}$
| $\neg\mathsf{Large(a)} \lor \mathsf{a} = \mathsf{c}$
| $\neg(\mathsf{c} = \mathsf{c} \land \mathsf{Tet(a)})$
|—
| $\mathsf{a} = \mathsf{b} \lor \mathsf{a} = \mathsf{c}$

5.18 ➶|✎

| $\mathsf{Cube(a)} \lor \mathsf{Tet(a)} \lor \mathsf{Large(a)}$
| $\neg\mathsf{Cube(a)} \lor \mathsf{a} = \mathsf{b} \lor \mathsf{Large(a)}$
| $\neg\mathsf{Large(a)} \lor \mathsf{a} = \mathsf{c}$
| $\neg(\mathsf{c} = \mathsf{c} \land \mathsf{Tet(a)})$
|—
| $\neg(\mathsf{Large(a)} \lor \mathsf{Tet(a)})$

5.19 ✎ Consider the following sentences.

1. *Folly was Claire's pet at 2 pm or at 2:05 pm.*
2. *Folly was Max's pet at 2 pm.*
3. *Folly was Claire's pet at 2:05 pm.*

Does (3) follow from (1) and (2)? Does (2) follow from (1) and (3)? Does (1) follow from (2) and (3)? In each case, give either a proof of consequence, or describe a situation that makes the premises true and the conclusion false. You may assume that Folly can only be one person's pet at any given time.

5.20 ✎ Suppose it is Friday night and you are going out with your boyfriend. He wants to see a romantic comedy, while you want to see the latest Wes Craven slasher movie. He points out that if he watches the Wes Craven movie, he will not be able to sleep because he can't stand the sight of blood, and he has to take the MCAT test tomorrow. If he does not do well on the MCAT, he won't get into medical school. Analyze your boyfriend's argument, pointing out where indirect proof is being used. How would you rebut his argument?

5.21 ✎ Describe an everyday example of an indirect proof that you have used in the last few days.

5.22 ✎⋆ Prove that indirect proof is a tautologically valid method of proof. That is, show that if $\mathsf{P}_1, \ldots, \mathsf{P}_n, \mathsf{S}$ is TT-contradictory, then $\neg\mathsf{S}$ is a tautological consequence of $\mathsf{P}_1, \ldots, \mathsf{P}_n$.

In the next three exercises we ask you to prove simple facts about the natural numbers. We do not expect you to phrase the proofs in FOL*. You will have to appeal to basic facts of arithmetic plus the definitions of even and odd number. This is OK, but make these appeals explicit. Also make explicit any use of proof by contradiction.*

5.23 ✎⋆ Assume that n^2 is odd. Prove that n is odd.

5.24 ✎⋆ Assume that $n + m$ is odd. Prove that $n \times m$ is even.

5.25 ✎⋆ Assume that n^2 is divisible by 3. Prove that n^2 is divisible by 9.

5.26 ✎⋆⋆ A good way to make sure you understand a proof is to try to generalize it. Prove that $\sqrt{3}$ is irrational. [Hint: You will need to figure out some facts about divisibility by 3 that parallel the facts we used about even and odd, for example, the fact expressed in Exercise 5.25.] Can you generalize these two results?

Section 5.4

Arguments with inconsistent premises

What follows from an inconsistent set of premises? If you look back at our definition of logical consequence, you will see that every sentence is a consequence of such a set. After all, if the premises are contradictory, then there are no circumstances in which they are all true. Thus, there are no circumstances in which the premises are true and the conclusion is false. Which is to say, in any situation in which the premises are all true (there aren't any of these!), the conclusion will be true as well. Hence any argument with an inconsistent set of premises is trivially valid. In particular, if one can establish a contradiction $\bot$ on the basis of the premises, then one is entitled to assert any sentence at all.

always valid

This often strikes students as a very odd method of reasoning, and for very good reason. For recall the distinction between a valid argument and a sound one. A *sound* argument is a valid argument with true premises. Even though any argument with an inconsistent set of premises is valid, no such argument is sound, since there is no way the premises of the argument can all be true. For this reason, an argument with an inconsistent set of premises is not worth *never sound*

much on its own. After all, the reason we are interested in logical consequence is because of its relation to truth. If the premises can't possibly be true, then even knowing that the argument is valid gives us no clue as to the truth or falsity of the conclusion. An unsound argument gives no more support for its conclusion than an invalid one.

In general, methods of proof don't allow us to show that an argument is unsound. After all, the truth or falsity of the premises is not a matter of logic, but of how the world happens to be. But in the case of arguments with inconsistent premises, our methods of proof do give us a way to show that at least one of the premises is false (though we might not know which one), and hence that the argument is unsound. To do this, we prove that the premises are inconsistent by deriving a contradiction.

Suppose, for example, you are given a proof that the following argument is valid:

$$\begin{array}{|l}
\mathsf{Home(max)} \lor \mathsf{Home(claire)} \\
\neg\mathsf{Home(max)} \\
\neg\mathsf{Home(claire)} \\
\hline
\mathsf{Home(max)} \land \mathsf{Happy(carl)}
\end{array}$$

While it is true that this conclusion is a consequence of the premises, your reaction should not be to believe the conclusion. Indeed, using proof by cases we can show that the premises are inconsistent, and hence that the argument is unsound. There is no reason to be convinced of the conclusion of an unsound argument.

Remember

A proof of a contradiction $\perp$ from premises $\mathsf{P}_1, \ldots, \mathsf{P}_n$ (without additional assumptions) shows that the premises are inconsistent. An argument with inconsistent premises is always valid, but more importantly, always unsound.

Exercises

5.27 ✎ Give two different proofs that the premises of the above argument are inconsistent. Your first should use proof by cases but not DeMorgan's law, while your second can use DeMorgan but not proof by cases.

You try it

1. Open the file Disjunction 2. The goal is to prove the sentence ◀

$$(\mathsf{Cube}(\mathsf{b}) \land \mathsf{Small}(\mathsf{b})) \lor (\mathsf{Cube}(\mathsf{b}) \land \mathsf{Large}(\mathsf{b}))$$

   The required proof is almost complete, though it may not look like it.

2. Focus on each empty step in succession, checking the step so that Fitch will fill in the default sentence. On the second empty step you will have to finish the sentence by typing in the second disjunct, (Cube(b) ∧ Large(b)), of the goal sentence. (If the last step does not generate a default, it is because you have not typed the right thing in the ∨ **Intro** step.) ◀

3. When you are finished, see if the proof checks out. Do you understand the proof? Could you have come up with it on your own? ◀

4. Save your completed proof as Proof Disjunction 2. ◀

. . . . ***Congratulations***

When you choose the ∨ **Intro** rule, and enter a disjunction at the focus step, you can use the **Add Support Steps** command to insert an appropriate support step. Fitch has to guess at the formula that you might want to cite as support. Fitch chooses the first disjunct, although any disjunct of the focus formula would be appropriate. **Add Support Steps** cannot be used with the ∨ **Elim** rule. When you use this rule, Fitch does not have enough information to fill in the support steps, even when you have given a formula at the focus step. You are on your own for this rule!

Exercises

6.1 If you skipped any of the **You try it** sections, go back and do them now. Submit the files Proof Conjunction 1, Proof Conjunction 2, Proof Conjunction 3, Proof Conjunction 4, Proof Disjunction 1, and Proof Disjunction 2.

6.2 Open the file Exercise 6.2, which contains an incomplete formal proof. As it stands, none of the steps check out, either because no rule has been specified, no support steps cited, or no sentence typed in. Provide the missing pieces and submit the completed proof.

Use Fitch to construct formal proofs for the following arguments. You will find Exercise *files for each argument in the usual place. As usual, name your solutions* Proof 6.x.

6.3

$$\begin{array}{|l} a = b \wedge b = c \wedge c = d \\ \hline a = c \wedge b = d \end{array}$$

6.4

$$\begin{array}{|l} (A \wedge B) \vee C \\ \hline C \vee B \end{array}$$

6.5

$$\begin{array}{|l} A \wedge (B \vee C) \\ \hline (A \wedge B) \vee (A \wedge C) \end{array}$$

6.6

$$\begin{array}{|l} (A \wedge B) \vee (A \wedge C) \\ \hline A \wedge (B \vee C) \end{array}$$

Section 6.3

Negation rules

Last but not least are the negation rules. It turns out that negation introduction is our most interesting and complex rule.

Negation elimination

The rule of negation elimination corresponds to a very trivial valid step, from $\neg\neg P$ to P. Schematically:

Negation Elimination (¬ Elim):

$$\begin{array}{r|l} & \neg\neg P \\ & \vdots \\ \triangleright & P \end{array}$$

Negation elimination gives us one direction of the principle of double negation. You might reasonably expect that our second negation rule, negation introduction, would simply give us the other direction. But if that's what you guessed, you guessed wrong.

Negation introduction

The rule of negation introduction corresponds to the method of indirect proof or proof by contradiction. Like $\vee$ **Elim**, it involves the use of a subproof, as will the formal analogs of all nontrivial methods of proof. The rule says that if you can prove a contradiction $\perp$ on the basis of an additional assumption P, then you are entitled to infer $\neg P$ from the original premises. Schematically:

the inserted sentence will be the negation of the assumption step of the cited subproof.

You try it

1. Open the file Negation 4. First look at the goal to see what sentence we are trying to prove. Then focus on each step in succession and check the step. Before moving to the next step, make sure you understand why the step checks out and, more important, why we are doing what we are doing at that step. At the empty steps, try to predict which sentence Fitch will provide as a default before you check the step. ◀

2. When you are done, make sure you understand the completed proof. Save your file as Proof Negation 4. ◀

Congratulations

Fitch will add a single support step if you use the **Add Support Steps** command when you have entered a formula and chosen the ¬ **Elim** rule. The support formula will be the formula from the focus step with two negation symbols preceding it. If you choose the ¬ **Intro** rule and use **Add Support Steps** then Fitch will insert a subproof as support, with the negation of the focus formula as the assumption of the subproof and ⊥ as the only other step in the subproof. You can also use **Add Support Steps** with ⊥ **Elim**. Whatever formula is present, Fitch inserts a single support step containing the support formula ⊥.

Exercises

6.7 If you skipped any of the **You try it** sections, go back and do them now. Submit the files Proof Negation 1, Proof Negation 2, Proof Negation 3, and Proof Negation 4.

6.8 (Substitution) In informal proofs, we allow you to substitute logically equivalent sentences for one another, even when they occur in the context of a larger sentence. For example, the following inference results from two uses of double negation, each applied to a part of the whole sentence:

$$\begin{array}{|l} \mathsf{P} \land (\mathsf{Q} \lor \neg\neg \mathsf{R}) \\ \hline \neg\neg \mathsf{P} \land (\mathsf{Q} \lor \mathsf{R}) \end{array}$$

How would we prove this using $\mathcal{F}$, which has no substitution rule? Open the file Exercise 6.8, which contains an incomplete formal proof of this argument. As it stands, none of the proof's steps check out, because no rules or support steps have been cited. Provide the missing justifications and submit the completed proof.

Evaluate each of the following arguments. If the argument is valid, use Fitch to give a formal proof using the rules you have learned. If it not valid, use Tarski's World to construct a counterexample world. In the last two proofs you will need to use **Ana Con** *to show that certain atomic sentences contradict one another to introduce $\perp$. Use* **Ana Con** *only in this way. That is, your use of* **Ana Con** *should cite exactly two atomic sentences in support of an introduction of $\perp$. If you have difficulty with any of these exercises, you may want to skip ahead and read Section 6.5.*

6.9

$$\begin{array}{|l} \mathsf{Cube}(b) \\ \neg(\mathsf{Cube}(c) \wedge \mathsf{Cube}(b)) \\ \hline \neg\mathsf{Cube}(c) \end{array}$$

6.10

$$\begin{array}{|l} \mathsf{Cube}(a) \vee \mathsf{Cube}(b) \\ \neg(\mathsf{Cube}(c) \wedge \mathsf{Cube}(b)) \\ \hline \neg\mathsf{Cube}(c) \end{array}$$

6.11

$$\begin{array}{|l} \mathsf{Dodec}(e) \\ \mathsf{Small}(e) \\ \neg\mathsf{Dodec}(e) \vee \mathsf{Dodec}(f) \vee \mathsf{Small}(e) \\ \hline \mathsf{Dodec}(f) \end{array}$$

6.12

$$\begin{array}{|l} \mathsf{Dodec}(e) \\ \neg\mathsf{Small}(e) \\ \neg\mathsf{Dodec}(e) \vee \mathsf{Dodec}(f) \vee \mathsf{Small}(e) \\ \hline \mathsf{Dodec}(f) \end{array}$$

6.13

$$\begin{array}{|l} \mathsf{Dodec}(e) \\ \mathsf{Large}(e) \\ \neg\mathsf{Dodec}(e) \vee \mathsf{Dodec}(f) \vee \mathsf{Small}(e) \\ \hline \mathsf{Dodec}(f) \end{array}$$

6.14

$$\begin{array}{|l} \mathsf{SameRow}(b,f) \vee \mathsf{SameRow}(c,f) \\ \quad\quad \vee\ \mathsf{SameRow}(d,f) \\ \neg\mathsf{SameRow}(c,f) \\ \mathsf{FrontOf}(b,f) \\ \neg(\mathsf{SameRow}(d,f) \wedge \mathsf{Cube}(f)) \\ \hline \neg\mathsf{Cube}(f) \end{array}$$

In the following two exercises, determine whether the sentences are consistent. If they are, use Tarski's World to build a world where the sentences are both true. If they are inconsistent, use Fitch to give a proof that they are inconsistent (that is, derive $\perp$ from them). You may use **Ana Con** *in your proof, but only applied to literals (that is, atomic sentences or negations of atomic sentences).*

6.15 $\neg(\mathsf{Larger}(a,b) \wedge \mathsf{Larger}(b,a))$
$\neg\mathsf{SameSize}(a,b)$

6.16 $\mathsf{Smaller}(a,b) \vee \mathsf{Smaller}(b,a)$
$\mathsf{SameSize}(a,b)$

Exercises

6.17 Try to recreate the following "proof" using Fitch.

1. $(\mathsf{Tet(a)} \land \mathsf{Large(c)}) \lor (\mathsf{Tet(a)} \land \mathsf{Dodec(b)})$	
2. $\mathsf{Tet(a)} \land \mathsf{Large(c)}$	
3. $\mathsf{Tet(a)}$	$\land$ **Elim**: 2
4. $\mathsf{Tet(a)} \land \mathsf{Dodec(b)}$	
5. $\mathsf{Dodec(b)}$	$\land$ **Elim**: 4
6. $\mathsf{Tet(a)}$	$\land$ **Elim**: 4
7. $\mathsf{Tet(a)}$	$\lor$ **Elim**: 1, 2–3, 4–6
8. $\mathsf{Tet(a)} \land \mathsf{Dodec(b)}$	$\land$ **Intro**: 7, 5

What step won't Fitch let you perform? Why? Is the conclusion a consequence of the premise? Discuss this example in the form of a clear English paragraph, and turn your paragraph in to your instructor.

Use Fitch to give formal proofs for the following arguments. You will need to use subproofs within subproofs to prove these.

6.18

$\mathsf{A} \lor \mathsf{B}$

$\mathsf{A} \lor \neg\neg\mathsf{B}$

6.19

$\mathsf{A} \lor \mathsf{B}$
$\neg\mathsf{B} \lor \mathsf{C}$

$\mathsf{A} \lor \mathsf{C}$

6.20

$\mathsf{A} \lor \mathsf{B}$
$\mathsf{A} \lor \mathsf{C}$

$\mathsf{A} \lor (\mathsf{B} \land \mathsf{C})$

Section 6.5

Strategy and tactics

Many students try constructing formal proofs by blindly piecing together a sequence of steps permitted by the introduction and elimination rules, a process no more related to reasoning than playing solitaire. This approach occasionally works, but more often than not it will fail—or at any rate, make it harder to find a proof. In this section, we will give you some advice about how to go about finding proofs when they don't jump right out at you. The advice consists of two important strategies and an essential maxim.

an important maxim

Here is the maxim: Always keep firmly in mind what the sentences in your proof mean! Students who pay attention to the meanings of the sentences avoid innumerable pitfalls, among them the pitfall of trying to prove a sentence that cannot find a counterexample, trying to find one often gives rise to insights about why the argument is valid, insights that can help you find the required proof.

We can summarize our strategy advice with a seven step procedure for approaching problems of this sort.

Remember

In assessing the validity of an argument, use the following method:

1. Understand what the sentences are saying.
2. Decide whether you think the conclusion follows from the premises.
3. If you think it does not follow, or are not sure, try to find a counterexample.
4. If you think it does follow, try to give an informal proof.
5. If a formal proof is called for, use the informal proof to guide you in finding one.
6. In giving consequence proofs, both formal and informal, don't forget the tactic of working backwards.
7. In working backwards, though, always check that your intermediate goals are consequences of the available information.

One final warning: One of the nice things about Fitch is that it will give you instant feedback about whether your proof is correct. This is a valuable learning tool, but it can be misused. You should not use Fitch as a crutch, trying out rule applications and letting Fitch tell you if they are correct. If you do this, then you are not really learning the system $\mathcal{F}$. One way to check up on yourself is to write a formal proof out on paper every now and then. If you try this and find you can't do it without Fitch's help, then you are using Fitch as a crutch, not a learning tool.

using Fitch as a crutch

Exercises

6.21 If you skipped the **You try it** section, go back and do it now. Submit the file Proof Strategy 1.

6.22 Give a formal proof mirroring the informal proof on page 137 of $\neg(b = c)$ from the premises $\mathsf{Cube}(c) \lor \mathsf{Dodec}(c)$ and $\mathsf{Tet}(b)$. You may apply **Ana Con** to literals in establishing $\perp$.

6.23 Give an informal proof that might have been used by the authors in constructing the formal proof shown on page 167.

In each of the following exercises, give an informal proof of the validity of the indicated argument. (You should never use the principle you are proving in your informal proof, for example in Exercise 6.24, you should not use DeMorgan in your informal proof.) Then use Fitch to construct a formal proof that mirrors your informal proof as much as possible. Turn in your informal proofs to your instructor and submit the formal proof in the usual way.

6.24

$\neg(A \lor B)$

$\neg A \land \neg B$

6.25

$\neg A \land \neg B$

$\neg(A \lor B)$

6.26

$A \lor (B \land C)$
$\neg B \lor \neg C \lor D$

$A \lor D$

6.27

$(A \land B) \lor (C \land D)$
$(B \land C) \lor (D \land E)$

$C \lor (A \land E)$

*In each of the following exercises, you should assess whether the argument is valid. If it is, use Fitch to construct a formal proof. You may use **Ana Con** but only involving literals and $\perp$. If it is not valid, use Tarski's World to construct a counterexample.*

6.28

$\mathsf{Cube}(c) \lor \mathsf{Small}(c)$
$\mathsf{Dodec}(c)$

$\mathsf{Small}(c)$

6.29

$\mathsf{Larger}(a, b) \lor \mathsf{Larger}(a, c)$
$\mathsf{Smaller}(b, a) \lor \neg\mathsf{Larger}(a, c)$

$\mathsf{Larger}(a, b)$

6.30

$\neg(\neg\mathsf{Cube}(a) \land \mathsf{Cube}(b))$
$\neg(\neg\mathsf{Cube}(b) \lor \mathsf{Cube}(c))$

$\mathsf{Cube}(a)$

6.31

$\mathsf{Dodec}(b) \lor \mathsf{Cube}(b)$
$\mathsf{Small}(b) \lor \mathsf{Medium}(b)$
$\neg(\mathsf{Small}(b) \land \mathsf{Cube}(b))$

$\mathsf{Medium}(b) \land \mathsf{Dodec}(b)$

6.32

$\mathsf{Dodec}(b) \lor \mathsf{Cube}(b)$
$\mathsf{Small}(b) \lor \mathsf{Medium}(b)$
$\neg\mathsf{Small}(b) \land \neg\mathsf{Cube}(b)$

$\mathsf{Medium}(b) \land \mathsf{Dodec}(b)$

SECTION 6.6

Proofs without premises

Not all proofs begin with the assumption of premises. This may seem odd, but in fact it is how we use our deductive system to show that a sentence is a logical truth. A sentence that can be proven without any premises at all is necessarily true. Here's a trivial example of such a proof, one that shows that $a = a \land b = b$ is a logical truth.

demonstrating logical truth

1. $a = a$	$=$ **Intro**
2. $b = b$	$=$ **Intro**
3. $a = a \land b = b$	$\land$ **Intro**: 1, 2

The first step of this proof is not a premise, but an application of **= Intro**. You might think that any proof without premises would have to start with this rule, since it is the only one that doesn't have to cite any supporting steps earlier in the proof. But in fact, this is not a very representative example of such proofs. A more typical and interesting proof without premises is the following, which shows that $\neg(P \land \neg P)$ is a logical truth.

1. $P \land \neg P$	
2. P	$\land$ **Elim**: 1
3. $\neg P$	$\land$ **Elim**: 1
4. $\bot$	$\bot$ **Intro**: 2, 3
5. $\neg(P \land \neg P)$	$\neg$ **Intro**: 1–4

Notice that there are no assumptions above the first horizontal Fitch bar, indicating that the main proof has no premises. The first step of the proof is the *subproof's* assumption. The subproof proceeds to derive a contradiction, based on this assumption, thus allowing us to conclude that the negation of the subproof's assumption follows without the need of premises. In other words, it is a logical truth.

When we want you to prove that a sentence is a logical truth, we will use Fitch notation to indicate that you must prove this without assuming any premises. For example the above proof shows that the following "argument" is valid:

$$\vdash \neg(P \land \neg P)$$

We close this section with the following reminder:

Remember

A proof without any premises shows that its conclusion is a logical truth.

Exercises

6.33 (Excluded Middle) Open the file Exercise 6.33. This contains an incomplete proof of the law of excluded middle, $P \lor \neg P$. As it stands, the proof does not check out because it's missing some sentences, some support citations, and some rules. Fill in the missing pieces and submit the completed proof as Proof 6.33. The proof shows that we can derive excluded middle in $\mathcal{F}$ without any premises.

In the following exercises, assess whether the indicated sentence is a logical truth in the blocks language. If so, use Fitch to construct a formal proof of the sentence from no premises (using **Ana Con** *if necessary, but only applied to literals). If not, use Tarski's World to construct a counterexample. (A counterexample here will simply be a world that makes the purported conclusion false.)*

6.34 $\vdash \neg(a = b \land \mathrm{Dodec}(a) \land \neg\mathrm{Dodec}(b))$

6.35 $\vdash \neg(a = b \land \mathrm{Dodec}(a) \land \mathrm{Cube}(b))$

6.36 $\vdash \neg(a = b \land b = c \land a \neq c)$

6.37 $\vdash \neg(a \neq b \land b \neq c \land a = c)$

6.38 $\vdash \neg(\mathrm{SameRow}(a, b) \land \mathrm{SameRow}(b, c) \land \mathrm{FrontOf}(c, a))$

6.39 $\vdash \neg(\mathrm{SameCol}(a, b) \land \mathrm{SameCol}(b, c) \land \mathrm{FrontOf}(a, c))$

The following sentences are all tautologies, and so should be provable in $\mathcal{F}$. Although the informal proofs are relatively simple, $\mathcal{F}$ makes fairly heavy going of them, since it forces us to prove even very obvious steps. Use Fitch to construct formal proofs. You may want to build on the proof of Excluded Middle given in Exercise 6.33. Alternatively, with the permission of your instructor, you may use **Taut Con***, but only to justify an instance of Excluded Middle. The Grade Grinder will indicate whether you used* **Taut Con** *or not.*

6.40 ✎★

$$\begin{array}{|l} \hline \mathsf{A} \lor \neg(\mathsf{A} \land \mathsf{B}) \end{array}$$

6.41 ✎★

$$\begin{array}{|l} \hline (\mathsf{A} \land \mathsf{B}) \lor \neg\mathsf{A} \lor \neg\mathsf{B} \end{array}$$

6.42 ✎★

$$\begin{array}{|l} \hline \neg\mathsf{A} \lor \neg(\neg\mathsf{B} \land (\neg\mathsf{A} \lor \mathsf{B})) \end{array}$$

Notice that the final column of this truth table is the same as that for $(P \rightarrow Q) \land (Q \rightarrow P)$. (See Exercise 7.3 below.) For this reason, logicians often treat a sentence of the form $P \leftrightarrow Q$ as an abbreviation of $(P \rightarrow Q) \land (Q \rightarrow P)$. Tarski's World also uses this abbreviation in the game. Thus, the game rule for $P \leftrightarrow Q$ is simple. Whenever a sentence of this form is encountered, it is replaced by $(P \rightarrow Q) \land (Q \rightarrow P)$.

game rule for $\leftrightarrow$

Remember

1. If P and Q are sentences of FOL, then so is $P \leftrightarrow Q$.
2. The sentence $P \leftrightarrow Q$ is true if and only if P and Q have the same truth value.

Exercises

For the following exercises, use Boole to determine whether the indicated pairs of sentences are tautologically equivalent. Feel free to have Boole build your reference columns and fill them out for you. Don't forget to indicate your assessment.

7.1 $A \rightarrow B$ and $\neg A \lor B$.

7.2 $\neg(A \rightarrow B)$ and $A \land \neg B$.

7.3 $A \leftrightarrow B$ and $(A \rightarrow B) \land (B \rightarrow A)$.

7.4 $A \leftrightarrow B$ and $(A \land B) \lor (\neg A \land \neg B)$.

7.5 $(A \land B) \rightarrow C$ and $A \rightarrow (B \lor C)$.

7.6 $(A \land B) \rightarrow C$ and $A \rightarrow (B \rightarrow C)$.

7.7 $A \rightarrow (B \rightarrow (C \rightarrow D))$ and $((A \rightarrow B) \rightarrow C) \rightarrow D$.

7.8 $A \leftrightarrow (B \leftrightarrow (C \leftrightarrow D))$ and $((A \leftrightarrow B) \leftrightarrow C) \leftrightarrow D$.

7.9 ✎ (Just in case) Prove that the ordinary (nonmathematical) use of *just in case* does not express a truth-functional connective. Use as your example the sentence *Max went home just in case Carl was hungry.*

7.10 (Evaluating sentences in a world) Using Tarski's World, run through Abelard's Sentences, evaluating them in Wittgenstein's World. If you make a mistake, play the game to see where you have gone wrong. Once you have gone through all the sentences, go back and make all the false ones true by changing one or more names used in the sentence. Submit your edited sentences as Sentences 7.10.

7.11 (Describing a world) Launch Tarski's World and choose **Hide Labels** from the **World** menu. Then, with the labels hidden, open Montague's World. In this world, each object has a name, and no object has more than one name. Start a new sentence file where you will describe some features of this world. Check each of your sentences to see that it is indeed a sentence and that it is true in this world.

1. Notice that if c is a tetrahedron, then a is not a tetrahedron. (Remember, in this world each object has exactly one name.) Use your first sentence to express this fact.
2. However, note that the same is true of b and d. That is, if b is a tetrahedron, then d isn't. Use your second sentence to express this.
3. Finally, observe that if b is a tetrahedron, then c isn't. Express this.
4. Notice that if a is a cube and b is a dodecahedron, then a is to the left of b. Use your next sentence to express this fact.
5. Use your next sentence to express the fact that if b and c are both cubes, then they are in the same row but not in the same column.
6. Use your next sentence to express the fact that b is a tetrahedron only if it is small. [Check this sentence carefully. If your sentence evaluates as false, then you've got the arrow pointing in the wrong direction.]
7. Next, express the fact that if a and d are both cubes, then one is to the left of the other. [Note: You will need to use a disjunction to express the fact that one is to the left of the other.]
8. Notice that d is a cube if and only if it is either medium or large. Express this.
9. Observe that if b is neither to the right nor left of d, then one of them is a tetrahedron. Express this observation.
10. Finally, express the fact that b and c are the same size if and only if one is a tetrahedron and the other is a dodecahedron.

Save your sentences as Sentences 7.11. Now choose **Show Labels** from the **World** menu. Verify that all of your sentences are indeed true. When verifying the first three, pay particular attention to the truth values of the various constituents. Notice that sometimes the conditional has a false antecedent and sometimes a true consequent. What it never has is a true antecedent and a false consequent. In each of these three cases, play the game committed to true. Make sure you understand why the game proceeds as it does.

7.12 (Translation) Translate the following English sentences into fol. Your translations will use all of the propositional connectives.

1. *If* ***a*** *is a tetrahedron then it is in front of* ***d****.*
2. ***a*** *is to the left of or right of* ***d*** *only if it's a cube.*
3. ***c*** *is between either* ***a*** *and* ***e*** *or* ***a*** *and* ***d****.*
4. ***c*** *is to the right of* ***a****, provided it (i.e.,* ***c****) is small.*

5. ***c*** *is to the right of* ***d*** *only if* ***b*** *is to the right of* ***c*** *and left of* ***e***.
6. *If* ***e*** *is a tetrahedron, then it's to the right of* ***b*** *if and only if it is also in front of* ***b***.
7. *If* ***b*** *is a dodecahedron, then if it isn't in front of* ***d*** *then it isn't in back of* ***d*** *either.*
8. ***c*** *is in back of* ***a*** *but in front of* ***e***.
9. ***e*** *is in front of* ***d*** *unless it (i.e.,* ***e****) is a large tetrahedron.*
10. *At least one of* ***a****,* ***c****, and* ***e*** *is a cube.*
11. ***a*** *is a tetrahedron only if it is in front of* ***b***.
12. ***b*** *is larger than both* ***a*** *and* ***e***.
13. ***a*** *and* ***e*** *are both larger than* ***c****, but neither is large.*
14. ***d*** *is the same shape as* ***b*** *only if they are the same size.*
15. ***a*** *is large if and only if it's a cube.*
16. ***b*** *is a cube unless* ***c*** *is a tetrahedron.*
17. *If* ***e*** *isn't a cube, either* ***b*** *or* ***d*** *is large.*
18. ***b*** *or* ***d*** *is a cube if either* ***a*** *or* ***c*** *is a tetrahedron.*
19. ***a*** *is large just in case* ***d*** *is small.*
20. ***a*** *is large just in case* ***e*** *is.*

Save your list of sentences as Sentences 7.12. Before submitting the file, you should complete Exercise 7.13.

7.13 (Checking your translations) Open Bolzano's World. Notice that all the English sentences from Exercise 7.12 are true in this world. Thus, if your translations are accurate, they will also be true in this world. Check to see that they are. If you made any mistakes, go back and fix them.

Remember that even if one of your sentences comes out true in Bolzano's World, it does not mean that it is a proper translation of the corresponding English sentence. If the translation is correct, it will have the same truth value as the English sentence in *every* world. So let's check your translations in some other worlds.

Open Wittgenstein's World. Here we see that the English sentences 3, 5, 9, 11, 12, 13, 14, and 20 are false, while the rest are true. Check to see that the same holds of your translations. If not, correct your translations (and make sure they are still true in Bolzano's World).

Next open Leibniz's World. Here half the English sentences are true (1, 2, 4, 6, 7, 10, 11, 14, 18, and 20) and half false (3, 5, 8, 9, 12, 13, 15, 16, 17, and 19). Check to see that the same holds of your translations. If not, correct your translations.

Finally, open Venn's World. In this world, all of the English sentences are false. Check to see that the same holds of your translations and correct them if necessary.

There is no need to submit any files for this exercise, but don't forget to submit Sentences 7.12.

7.14 (Figuring out sizes and shapes) Open Euler's Sentences. The nine sentences in this file uniquely determine the shapes and sizes of blocks a, b, and c. See if you can figure out the solution just by thinking about what the sentences mean and using the informal methods of proof you've already studied. When you've figured it out, submit a world in which all of the sentences are true.

7.15 (More sizes and shapes) Start a new sentence file and use it to translate the following English sentences.

1. *If **a** is a tetrahedron, then **b** is also a tetrahedron.*
2. ***c** is a tetrahedron if **b** is.*
3. ***a** and **c** are both tetrahedra only if at least one of them is large.*
4. ***a** is a tetrahedron but **c** isn't large.*
5. *If **c** is small and **d** is a dodecahedron, then **d** is neither large nor small.*
6. ***c** is medium only if none of **d**, **e**, and **f** are cubes.*
7. ***d** is a small dodecahedron unless **a** is small.*
8. ***e** is large just in case it is a fact that **d** is large if and only if **f** is.*
9. ***d** and **e** are the same size.*
10. ***d** and **e** are the same shape.*
11. ***f** is either a cube or a dodecahedron, if it is large.*
12. ***c** is larger than **e** only if **b** is larger than **c**.*

Save these sentences as Sentences 7.15. Then see if you can figure out the sizes and shapes of a, b, c, d, e, and f. You will find it helpful to approach this problem systematically, filling in the following table as you reason about the sentences:

	a	***b***	***c***	***d***	***e***	***f***
Shape:						
Size:						

When you have filled in the table, use it to guide you in building a world in which the twelve English sentences are true. Verify that your translations are true in this world as well. Submit both your sentence file and your world file.

7.16 (Name that object) Open Sherlock's World and Sherlock's Sentences. You will notice that none of the objects in this world has a name. Your task is to assign the names a, b, and c in such a way that all the sentences in the list come out true. Submit the modified world as World 7.16.

7.17 (Building a world) Open Boolos' Sentences. Submit a world in which all five sentences in this file are true.

7.18 Using the symbols introduced in Table 1.2, page 30, translate the following sentences into FOL. Submit your translations as a sentence file.

1. *If Claire gave Folly to Max at 2:03 then Folly belonged to her at 2:00 and to him at 2:05.*
2. *Max fed Folly at 2:00 pm, but if he gave her to Claire then, Folly was not hungry five minutes later.*
3. *If neither Max nor Claire fed Folly at 2:00, then she was hungry.*
4. *Max was angry at 2:05 only if Claire fed either Folly or Scruffy five minutes before.*
5. *Max is a student if and only if Claire is not.*

7.19 Using Table 1.2 on page 30, translate the following into colloquial English.

1. $(\mathsf{Fed}(\mathsf{max}, \mathsf{folly}, 2{:}00) \lor \mathsf{Fed}(\mathsf{claire}, \mathsf{folly}, 2{:}00)) \to \mathsf{Pet}(\mathsf{folly})$
2. $\mathsf{Fed}(\mathsf{max}, \mathsf{folly}, 2{:}30) \leftrightarrow \mathsf{Fed}(\mathsf{claire}, \mathsf{scruffy}, 2{:}00)$
3. $\neg\mathsf{Hungry}(\mathsf{folly}, 2{:}00) \to \mathsf{Hungry}(\mathsf{scruffy}, 2{:}00)$
4. $\neg(\mathsf{Hungry}(\mathsf{folly}, 2{:}00) \to \mathsf{Hungry}(\mathsf{scruffy}, 2{:}00))$

7.20 Translate the following into FOL as best you can. Explain any predicates and function symbols you use, and any shortcomings in your first-order translations.

1. *If Abe can fool Stephen, surely he can fool Ulysses.*
2. *If you scratch my back, I'll scratch yours.*
3. *France will sign the treaty only if Germany does.*
4. *If Tweedledee gets a party, so will Tweedledum, and vice versa.*
5. *If John and Mary went to the concert together, they must like each other.*

7.21 (The monkey principle) One of the stranger uses of *if...then...* in English is as a roundabout way to express negation. Suppose a friend of yours says *If Keanu Reeves is a great actor, then I'm a monkey's uncle.* This is simply a way of denying the antecedent of the conditional, in this case that Keanu Reeves is a great actor. Explain why this works. Your explanation should appeal to the truth table for $\to$, but it will have to go beyond that. Turn in your explanation and also submit a Boole table showing that $\mathsf{A} \to \bot$ is equivalent to $\neg\mathsf{A}$.

Section 7.3

Conversational implicature

In translating from English to FOL, there are many problematic cases. For example, many students resist translating a sentence like *Max is home unless Claire is at the library* as:

$$\neg\mathsf{Library}(\mathsf{claire}) \to \mathsf{Home}(\mathsf{max})$$

Exercises

7.22 ✎ Suppose Claire asserts the sentence *Max managed to get Carl home.* Does this logically imply, or just conversationally implicate, that it was hard to get Carl home? Justify your answer.

7.23 ✎ Suppose Max asserts the sentence *We can walk to the movie or we can drive.* Does his assertion logically imply, or merely implicate, that we cannot both walk and drive? How does this differ from the soup or salad example?

7.24 ✎⋆ Consider the sentence *Max is home in spite of the fact that Claire is at the library.* What would be the best translation of this sentence into FOL? Clearly, whether you would be inclined to use this sentence is not determined simply by the truth values of the atomic sentences *Max is home* and *Claire is at the library.* This may be because *in spite of the fact* is, like *because*, a non-truth-functional connective, or because it carries, like *but*, additional conversational implicatures. (See our discussion of *because* earlier in this chapter and the discussion of *but* in Chapter 3.) Which explanation do you think is right? Justify your answer.

Section 7.4

Truth-functional completeness

We now have at our disposal five truth-functional connectives, one unary ($\neg$), and four binary ($\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$). Should we introduce any more? Though we've seen a few English expressions that can't be expressed in FOL, like *because*, these have not been truth functional. We've also run into others, like *neither... nor...*, that *are* truth functional, but which we can easily express using the existing connectives of FOL.

The question we will address in the current section is whether there are any truth-functional connectives that we need to add to our language. Is it possible that we might encounter an English construction that is truth functional but which we cannot express using the symbols we have introduced so far? If so, this would be an unfortunate limitation of our language.

How can we possibly answer this question? Well, let's begin by thinking about binary connectives, those that apply to two sentences to make a third. How many binary truth-functional connectives are possible? If we think about the possible truth tables for such connectives, we can compute the total number. First, since we are dealing with binary connectives, there are four rows in each table. Each row can be assigned either TRUE or FALSE, so there are $2^4 = 16$ ways of doing this. For example, here is the table that captures the truth function expressed by *neither... nor...*.

one use of $\vee$ **Elim**, and one use of $\neg$ **Intro** (see Exercise 7.26). This is why we haven't skimped on connectives.

Remember

1. A set of connectives is *truth-functionally complete* if the connectives allow us to express every truth function.
2. Various sets of connectives, including the Boolean connectives, are truth-functionally complete.

Exercises

7.25 (Replacing $\wedge$, $\rightarrow$, and $\leftrightarrow$) Use Tarski's World to open the file Sheffer's Sentences. In this file, you will find the following sentences in the odd-numbered positions:

1. $\mathsf{Tet(a)} \wedge \mathsf{Small(a)}$
3. $\mathsf{Tet(a)} \rightarrow \mathsf{Small(a)}$
5. $\mathsf{Tet(a)} \leftrightarrow \mathsf{Small(a)}$
7. $(\mathsf{Cube(b)} \wedge \mathsf{Cube(c)}) \rightarrow (\mathsf{Small(b)} \leftrightarrow \mathsf{Small(c)})$

In each even-numbered slot, enter a sentence that is equivalent to the one above it, but which uses only the connectives $\neg$ and $\vee$. Before submitting your solution file, you might want to try out your sentences in several worlds to make sure the new sentences have the expected truth values.

7.26 (Basic versus defined symbols in proofs) Treating a symbol as basic, with its own rules, or as a defined symbol, without its own rules, makes a big difference to the complexity of proofs. Use Fitch to open the file Exercise 7.26. In this file, you are asked to construct a proof of $\neg(\neg\mathsf{A} \vee \neg\mathsf{B})$ from the premises A and B. A proof of the equivalent sentence $\mathsf{A} \wedge \mathsf{B}$ would of course take a single step.

7.27 (Simplifying *if. . . then. . . else*) Assume that P, Q, and R are atomic sentences. See if you can simplify the sentence we came up with to express $\clubsuit(\mathsf{P}, \mathsf{Q}, \mathsf{R})$ (*if P then Q, else R*), so that it becomes a disjunction of two sentences, each of which is a conjunction of two literals. Submit your solution as a Tarski's World sentence file.

7.28 (Expressing another ternary connective) Start a new sentence file using Tarski's World. Use the method we have developed to express the ternary connective ♡ defined in the following truth table, and enter this as the first sentence in your file. Then see if you can simplify the result as much as possible. Enter the simplified form as your second sentence. (This sentence should have no more than two occurrences each of P, Q, and R, and no more than six occurrences of the Boolean connectives, ∨, ∧ and ¬.)

P	Q	R	♡(P, Q, R)
T	T	T	**T**
T	T	F	**T**
T	F	T	**T**
T	F	F	**F**
F	T	T	**F**
F	T	F	**T**
F	F	T	**T**
F	F	F	**T**

7.29 (Sheffer stroke) Another binary connective that is truth-functionally complete on its own is called the Sheffer stroke, named after H. M. Sheffer, one of the logicians who discovered and studied it. It is also known as *nand* by analogy with *nor*. Here is its truth table:

P	Q	P | Q
T	T	**F**
T	F	**T**
F	T	**T**
F	F	**T**

Show how to express ¬P, P ∧ Q, and P ∨ Q using the Sheffer stroke. (We remind you that nowadays, the symbol | has been appropriated as an alternative for ∨. Don't let that confuse you.)

7.30 (Putting monkeys to work) Suppose we have the single binary connective →, plus the symbol for absurdity ⊥. Using just these expressions, see if you can find a way to express ¬P, P ∧ Q, and P ∨ Q. [Hint: Don't forget what you learned in Exercise 7.21.]

7.31 (Another non-truth-functional connective) Show that truth value at a particular time of the sentence *Max is home whenever Claire is at the library* is not determined by the truth values of the atomic sentences *Max is home* and *Claire is at the library* at that same time. That is, show that *whenever* is not truth functional.

7.32 (Exclusive disjunction) Suppose we had introduced $\triangledown$ to express exclusive disjunction. Is the following a valid method of proof for this connective?

$$
\begin{array}{l|l}
 & \mathsf{P} \triangledown \mathsf{Q} \\
 & \quad \begin{array}{|l} \mathsf{P} \\ \hline \vdots \\ \mathsf{S} \end{array} \\
 & \quad \begin{array}{|l} \mathsf{Q} \\ \hline \vdots \\ \mathsf{T} \end{array} \\
\triangleright & \mathsf{S} \triangledown \mathsf{T}
\end{array}
$$

If you say *yes*, justify your answer; if *no*, give an example where the method sanctions an invalid inference.

State valid introduction and elimination rules for $\triangledown$ using the same format we use to state the introduction and elimination rules of $\mathcal{F}$. You may need more than one of each.

Section 7.5

Alternative notation

As with the other truth-functional connectives, there are alternative notations for the material conditional and biconditional. The most common alternative to $\mathsf{P} \rightarrow \mathsf{Q}$ is $\mathsf{P} \supset \mathsf{Q}$. Polish notation for the conditional is Cpq. The most common alternative to $\mathsf{P} \leftrightarrow \mathsf{Q}$ is $\mathsf{P} \equiv \mathsf{Q}$. The Polish notation for the biconditional is Epq.

Remember

The following table summarizes the alternative notations discussed so far.

Our notation	**Common equivalents**
$\neg\mathsf{P}$	$\sim \mathsf{P}$, $\overline{\mathsf{P}}$, !P, Np
$\mathsf{P} \wedge \mathsf{Q}$	P&Q, P&&Q, $\mathsf{P} \cdot \mathsf{Q}$, PQ, Kpq
$\mathsf{P} \vee \mathsf{Q}$	$\mathsf{P} \mid \mathsf{Q}$, $\mathsf{P} \parallel \mathsf{Q}$, Apq
$\mathsf{P} \rightarrow \mathsf{Q}$	$\mathsf{P} \supset \mathsf{Q}$, Cpq
$\mathsf{P} \leftrightarrow \mathsf{Q}$	$\mathsf{P} \equiv \mathsf{Q}$, Epq

it is divisible by 2, so we have (3) $\rightarrow$ (2). Next, we prove (2) $\rightarrow$ (1) by proving its contrapositive. Thus, assume n is odd and prove n^2 is odd. Since n is odd, we can write it in the form $2m + 1$. But then (as we've already shown) $n^2 = 2(2m^2 + 2m) + 1$ which is also odd. Finally, let us prove (1) $\rightarrow$ (3). If n is even, it can be expressed as $2m$. Thus, $n^2 = (2m)^2 = 4m^2$, which is divisible by 4. This completes the cycle, showing that the three conditions are indeed equivalent.

When you apply this method, you should look for simple or obvious implications, like (1) $\rightarrow$ (3) above, or implications that you've already established, like (2) $\rightarrow$ (1) above, and try to build them into your cycle of conditionals.

Remember

1. The method of conditional proof: To prove $\mathsf{P} \rightarrow \mathsf{Q}$, assume P and prove Q.
2. To prove a number of biconditionals, try to arrange them into a cycle of conditionals.

Exercises

8.1 ✎ In the following list we give a number of inference patterns, some of which are valid, some invalid. For each pattern, decide whether you think it is valid and say so. Later, we will return to these patterns and ask you to give formal proofs for the valid ones and counterexamples for the invalid ones. But for now, just assess their validity.

1. *Affirming the Consequent:* From $\mathsf{A} \rightarrow \mathsf{B}$ and B, infer A.
2. *Modus Tollens:* From $\mathsf{A} \rightarrow \mathsf{B}$ and $\neg\mathsf{B}$, infer $\neg\mathsf{A}$.
3. *Strengthening the Antecedent:* From $\mathsf{B} \rightarrow \mathsf{C}$, infer $(\mathsf{A} \land \mathsf{B}) \rightarrow \mathsf{C}$.
4. *Weakening the Antecedent:* From $\mathsf{B} \rightarrow \mathsf{C}$, infer $(\mathsf{A} \lor \mathsf{B}) \rightarrow \mathsf{C}$.
5. *Strengthening the Consequent:* From $\mathsf{A} \rightarrow \mathsf{B}$, infer $\mathsf{A} \rightarrow (\mathsf{B} \land \mathsf{C})$.
6. *Weakening the Consequent:* From $\mathsf{A} \rightarrow \mathsf{B}$, infer $\mathsf{A} \rightarrow (\mathsf{B} \lor \mathsf{C})$.
7. *Constructive Dilemma:* From $\mathsf{A} \lor \mathsf{B}$, $\mathsf{A} \rightarrow \mathsf{C}$, and $\mathsf{B} \rightarrow \mathsf{D}$, infer $\mathsf{C} \lor \mathsf{D}$.
8. *Transitivity of the Biconditional:* From $\mathsf{A} \leftrightarrow \mathsf{B}$ and $\mathsf{B} \leftrightarrow \mathsf{C}$, infer $\mathsf{A} \leftrightarrow \mathsf{C}$.

8.2 ➶|✎ Open Conditional Sentences. Suppose that the sentences in this file are your premises. Now consider the five sentences listed below. Some of these sentences are consequences of these premises, some are not. For those that are consequences, give informal proofs and turn them in to your instructor. For those that are not consequences, submit counterexample worlds in which the premises are true but the conclusion false. Name the counterexamples World 8.2.x, where x is the number of the sentence.

1. Tet(e)
2. Tet(c) → Tet(e)
3. Tet(c) → Larger(f, e)
4. Tet(c) → LeftOf(c, f)
5. Dodec(e) → Smaller(e, f)

The following arguments are all valid. Turn in informal proofs of their validity. You may find it helpful to translate the arguments into FOL *before trying to give proofs, though that's not required. Explicitly note any inferences using modus ponens, biconditional elimination, or conditional proof.*

8.3 ✎

The unicorn, if it is not mythical, is a mammal, but if it is mythical, it is immortal.
If the unicorn is either immortal or a mammal, it is horned.
The unicorn, if horned, is magical.

The unicorn is magical.

8.4 ✎

The unicorn, if horned, is elusive and dangerous.
If elusive or mythical, the unicorn is rare.
If a mammal, the unicorn is not rare.

The unicorn, if horned, is not a mammal.

8.5 ✎

The unicorn, if horned, is elusive and magical, but if not horned, it is neither.
If the unicorn is not horned, it is not mythical.

The unicorn is horned if and only if magical or mythical.

8.6 ✎

a is a large tetrahedron or a small cube.
b is not small.
If *a* is a tetrahedron or a cube, then *b* is large or small.
a is a tetrahedron only if *b* is medium.
---
a is small and *b* is large.

8.7 ✎

b is small unless it's a cube.
If *c* is small, then either *d* or *e* is too.
If *d* is small, then *c* is not.
If *b* is a cube, then *e* is not small.

If *c* is small, then so is *b*.

8.8 ✎

d is in the same row as *a*, *b* or *c*.
If *d* is in the same row as *b*, then it is in the same row as *a* only if it's not in the same row as *c*.
d is in the same row as *a* if and only if it is in the same row as *c*.
---
d is in the same row as *a* if and only if it is not in the same row as *b*.

8.9 ✎

a is either a cube, a dodecahedron, or a tetrahedron.
a is small, medium, or large.
a is medium if and only if it's a dodecahedron.
a is a tetrahedron if and only if it is large.

---

a is a cube if and only if it's small.

8.10 ➶|✎ Open Between Sentences. Determine whether this set of sentences is satisfiable or not. If it is, submit a world in which all the sentences are true. If not, give an informal proof that the sentences are inconsistent. That is, assume all of them and derive a contradiction.

8.11 ✎ Analyze the structure of the informal proof in support of the following claim: *If the U.S. does not cut back on its use of oil soon, parts of California will be flooded within 50 years.* Are there weak points in the argument? What premises are implicitly assumed in the proof? Are they plausible?

> **Proof:** Suppose the U.S. does not cut back on its oil use soon. Then it will be unable to reduce its carbon dioxide emissions substantially in the next few years. But then the countries of China, India and Brazil will refuse to join in efforts to curb carbon dioxide emissions. As these countries develop without such efforts, the emission of carbon dioxide will get much worse, and so the greenhouse effect will accelerate. As a result the sea will get warmer, ice will melt, and the sea level will rise. In which case, low lying coastal areas in California will be subject to flooding within 50 years. So if we do not cut back on our oil use, parts of California will be flooded within 50 years.

8.12 ✎ Describe an everyday example of reasoning that uses the method of conditional proof.

8.13 ✎⋆ Prove: $\mathsf{Odd}(\mathsf{n} + \mathsf{m}) \to \mathsf{Even}(\mathsf{n} \times \mathsf{m})$.
[Hint: Compare this with Exercise 5.24 on page 141.]

8.14 ✎⋆ Prove: $\mathsf{Irrational}(\mathsf{x}) \to \mathsf{Irrational}(\sqrt{\mathsf{x}})$.
[Hint: It is easier to prove the contrapositive.]

8.15 ✎⋆ Prove that the following conditions on the natural number n are all equivalent. Use as few conditional proofs as possible.

1. n is divisible by 3
2. n^2 is divisible by 3
3. n^2 is divisible by 9
4. n^3 is divisible by 3
5. n^3 is divisible by 9
6. n^3 is divisible by 27

8.16 ✎⋆ Give an informal proof that if R is a tautological consequence of $\mathsf{P}_1, \ldots, \mathsf{P}_n$ and Q, then $\mathsf{Q} \to \mathsf{R}$ is a tautological consequence of $\mathsf{P}_1, \ldots, \mathsf{P}_n$.

6. When you are done, save your proof as Proof Conditional 3. ◀

. . . . *Congratulations*

The default and generous uses of the biconditional rules are exactly like those for the conditional connective, and **Add Support Steps** works exactly as you would expect.

Exercises

8.17 If you skipped any of the **You try it** sections, go back and do them now. Submit the files Proof Conditional 1, Proof Conditional 2, and Proof Conditional 3.

In the following exercises we return to the patterns of inference discussed in Exercise 8.1. Some of these are valid, some invalid. For each valid pattern, construct a formal proof in Fitch. For each invalid pattern, give a counterexample using Tarski's World. To give a counterexample in these cases, you will have to come up with sentences of the blocks language that fit the pattern, and a world that makes those specific premises true and the conclusion false. Submit both the world and the sentence file. In the sentence file, list the premises first and the conclusion last.

8.18 *Affirming the Consequent:*
From $A \rightarrow B$ and B, infer A.

8.19 *Modus Tollens:*
From $A \rightarrow B$ and $\neg B$, infer $\neg A$.

8.20 *Strengthening the Antecedent:*
From $B \rightarrow C$, infer $(A \wedge B) \rightarrow C$.

8.21 *Weakening the Antecedent:*
From $B \rightarrow C$, infer $(A \vee B) \rightarrow C$.

8.22 *Strengthening the Consequent:*
From $A \rightarrow B$, infer $A \rightarrow (B \wedge C)$.

8.23 *Weakening the Consequent:*
From $A \rightarrow B$, infer $A \rightarrow (B \vee C)$.

8.24 *Constructive Dilemma:*
From $A \vee B$, $A \rightarrow C$, and $B \rightarrow D$, infer $C \vee D$.

8.25 *Transitivity of the Biconditional:*
From $A \leftrightarrow B$ and $B \leftrightarrow C$, infer $A \leftrightarrow C$.

Use Fitch to construct formal proofs for the following arguments. In two cases, you may find yourself re-proving an instance of the law of Excluded Middle, $P \vee \neg P$, in order to complete your proof. If you've forgotten how to do that, look back at your solution to Exercise 6.33. Alternatively, with the permission of your instructor, you may use **Taut Con** *to justify an instance of Excluded Middle.*

8.26

$P \rightarrow (Q \rightarrow P)$

8.27

$(P \rightarrow (Q \rightarrow R)) \leftrightarrow ((P \wedge Q) \rightarrow R)$

8.28

$$\begin{array}{|l} \mathsf{P} \leftrightarrow \neg\mathsf{P} \\ \hline \bot \end{array}$$

8.29

$$\begin{array}{|l} \\ \hline (\mathsf{P} \to \mathsf{Q}) \leftrightarrow (\neg\mathsf{P} \lor \mathsf{Q}) \end{array}$$

8.30

$$\begin{array}{|l} \\ \hline \neg(\mathsf{P} \to \mathsf{Q}) \leftrightarrow (\mathsf{P} \land \neg\mathsf{Q}) \end{array}$$

The following arguments are translations of those given in Exercises 8.3–8.9. (For simplicity we have assumed "the unicorn" refers to a specific unicorn named Charlie. This is less than ideal, but the best we can do without quantifiers.) Use Fitch to formalize the proofs you gave of their validity. You will need to use **Ana Con** *to introduce* $\bot$ *in two of your proofs.*

8.31

$$\begin{array}{|l} (\neg\mathsf{Mythical(c)} \to \mathsf{Mammal(c)}) \\ \quad \land (\mathsf{Mythical(c)} \to \neg\mathsf{Mortal(c)}) \\ (\neg\mathsf{Mortal(c)} \lor \mathsf{Mammal(c)}) \to \mathsf{Horned(c)} \\ \mathsf{Horned(c)} \to \mathsf{Magical(c)} \\ \hline \mathsf{Magical(c)} \end{array}$$

8.32

$$\begin{array}{|l} \mathsf{Horned(c)} \to (\mathsf{Elusive(c)} \\ \quad \land \mathsf{Dangerous(c)}) \\ (\mathsf{Elusive(c)} \lor \mathsf{Mythical(c)}) \to \mathsf{Rare(c)} \\ \mathsf{Mammal(c)} \to \neg\mathsf{Rare(c)} \\ \hline \mathsf{Horned(c)} \to \neg\mathsf{Mammal(c)} \end{array}$$

8.33

$$\begin{array}{|l} (\mathsf{Horned(c)} \to (\mathsf{Elusive(c)} \land \mathsf{Magical(c)})) \\ \quad \land (\neg\mathsf{Horned(c)} \to (\neg\mathsf{Elusive(c)} \\ \quad \land \neg\mathsf{Magical(c)})) \\ \neg\mathsf{Horned(c)} \to \neg\mathsf{Mythical(c)} \\ \hline \mathsf{Horned(c)} \leftrightarrow (\mathsf{Magical(c)} \lor \mathsf{Mythical(c)}) \end{array}$$

8.34

$$\begin{array}{|l} (\mathsf{Tet(a)} \land \mathsf{Large(a)}) \lor (\mathsf{Cube(a)} \\ \quad \land \mathsf{Small(a)}) \\ \neg\mathsf{Small(b)} \\ (\mathsf{Tet(a)} \lor \mathsf{Cube(a)}) \to (\mathsf{Large(b)} \\ \quad \lor \mathsf{Small(b)}) \\ \mathsf{Tet(a)} \to \mathsf{Medium(b)} \\ \hline \mathsf{Small(a)} \land \mathsf{Large(b)} \end{array}$$

8.35

$$\begin{array}{|l} \neg\mathsf{Cube(b)} \to \mathsf{Small(b)} \\ \mathsf{Small(c)} \to (\mathsf{Small(d)} \lor \mathsf{Small(e)}) \\ \mathsf{Small(d)} \to \neg\mathsf{Small(c)} \\ \mathsf{Cube(b)} \to \neg\mathsf{Small(e)} \\ \hline \mathsf{Small(c)} \to \mathsf{Small(b)} \end{array}$$

8.36

$$\begin{array}{|l} \mathsf{SameRow(d,a)} \lor \mathsf{SameRow(d,b)} \\ \quad \lor \mathsf{SameRow(d,c)} \\ \mathsf{SameRow(d,b)} \to (\mathsf{SameRow(d,a)} \\ \quad \to \neg\mathsf{SameRow(d,c)}) \\ \mathsf{SameRow(d,a)} \leftrightarrow \mathsf{SameRow(d,c)} \\ \hline \mathsf{SameRow(d,a)} \leftrightarrow \neg\mathsf{SameRow(d,b)} \end{array}$$

8.37

$$
\begin{array}{|l}
\mathsf{Cube(a)} \lor \mathsf{Dodec(a)} \lor \mathsf{Tet(a)} \\
\mathsf{Small(a)} \lor \mathsf{Medium(a)} \lor \mathsf{Large(a)} \\
\mathsf{Medium(a)} \leftrightarrow \mathsf{Dodec(a)} \\
\mathsf{Tet(a)} \leftrightarrow \mathsf{Large(a)} \\
\hline
\mathsf{Cube(a)} \leftrightarrow \mathsf{Small(a)}
\end{array}
$$

8.38 Use Fitch to give formal proofs of both $(\mathsf{P} \land \mathsf{Q}) \rightarrow \mathsf{P}$ and the equivalent sentence $\neg(\mathsf{P} \land \mathsf{Q}) \lor \mathsf{P}$. (You will find the exercise files in Exercise 8.38.1 and Exercise 8.38.2.) Do you see why it is convenient to include $\rightarrow$ in FOL, rather than define it in terms of the Boolean connectives?

Section 8.3

Soundness and completeness

We have now introduced formal rules for all of our truth-functional connectives. Let's step back for a minute and ask two important questions about the formal system $\mathcal{F}$. The questions get at two desirable properties of a deductive system, which logicians call *soundness* and *completeness.* Don't be confused by the names, however. These uses of *sound* and *complete* are different from their use in the notions of a sound argument and a truth-functionally complete set of connectives.

Soundness

We intend our formal system $\mathcal{F}$ to be a correct system of deduction in the sense that any argument that can be proven valid in $\mathcal{F}$ should be genuinely valid. The first question that we will ask, then, is whether we have succeeded in this goal. Does the system $\mathcal{F}$ allow us to construct proofs only of genuinely valid arguments? This is known as the soundness question for the deductive system $\mathcal{F}$.

soundness of a deductive system

The answer to this question may seem obvious, but it deserves a closer look. After all, consider the rule of inference suggested in Exercise 7.32 on page 198. Probably, when you first looked at this rule, it seemed pretty reasonable, even though on closer inspection you realized it was not (or maybe you got the problem wrong). How can we be sure that something similar might not be the case for one of our official rules? Maybe there is a flaw in one of them but we just haven't thought long enough or hard enough to discover it.

Or maybe there are problems that go beyond the individual rules, something about the way the rules interact. Consider for example the following

Exercises

Decide whether the following two arguments are provable in $\mathcal{F}_T$ without actually trying to find proofs. Do this by constructing a truth table in Boole to assess their tautological validity. Submit the table. Then explain clearly how you know the argument is or is not provable by applying the Soundness and Completeness results. Turn in your explanations to your instructor. (The explanations are more important than the tables, so don't forget the second part!)

8.39 ➶|✎

| A ∧ (B ∨ ¬A ∨ (C ∧ D))
| E ∧ (D ∨ ¬(A ∧ (B ∨ D)))
|—
| A ∧ B

8.40 ➶|✎

| A ∧ (B ∨ ¬A ∨ (C ∧ D)) ∧ ¬(A ∧ D)
|—
| ¬(E ∧ (D ∨ ¬(A ∧ (B ∨ D))))

In the proof of the Soundness Theorem, we only treated three of the twelve rules of $\mathcal{F}_T$. The next three problems ask you to treat some of the other rules.

8.41 ✎⋆ Give the argument required for the ¬ **Elim** case of the Soundness proof. Your argument will be very similar to the one we gave for → **Elim**.

8.42 ✎⋆⋆ Give the argument required for the ¬ **Intro** case of the Soundness proof. Your argument will be similar to the one we gave for → **Intro**.

8.43 ✎⋆⋆ Give the argument required for the ∨ **Elim** case of the Soundness proof.

Section 8.4

Valid arguments: some review exercises

There is wisdom in the old saying "Don't lose sight of the forest for the trees." The forest in our case is an understanding of valid arguments. The trees are the various methods of proofs, formal and informal, and the notions of counterexample, tautology, and the like. The problems in this section are intended to remind you of the relationship between the forest and the trees, as well as to help you review the main ideas discussed so far.

Since you now know that our introduction and elimination rules suffice to prove any tautologically valid argument, you should feel free to use **Taut Con** in doing these exercises. In fact, you may use it in your formal proofs from now on, but with this important proviso: Make sure that you use it only in cases where the inference step is obvious and would go by without notice in an informal proof. For example, you may use it to introduce the law of excluded middle or to apply a DeMorgan equivalence. But you should still use rules like $\lor$ **Elim**, $\neg$ **Intro**, and $\rightarrow$ **Intro** when your informal proof would use proof by cases, proof by contradiction, or conditional proof. Any one-step proofs that consist of a single application of **Taut Con** will be counted as wrong!

Before doing these problems, go back and read the material in the **Remember** boxes, paying special attention to the strategy for evaluating arguments on page 173.

Remember

From this point on in the book, you may use **Taut Con** in formal proofs, but only to skip simple steps that would go unmentioned in an informal proof.

Exercises

In the following exercises, you are given arguments in the blocks language. Evaluate each argument's validity. If it is valid, construct a formal proof to show this. If you need to use **Ana Con**, *use it only to derive $\bot$ from atomic sentences. If the argument is invalid, you should use Tarski's World to construct a counterexample world.*

8.44

$$\begin{array}{|l} \mathsf{Adjoins}(a,b) \land \mathsf{Adjoins}(b,c) \\ \mathsf{SameRow}(a,c) \\ \hline a \neq c \end{array}$$

8.45

$$\begin{array}{|l} \\ \hline \neg(\mathsf{Cube}(b) \land b = c) \lor \mathsf{Cube}(c) \end{array}$$

8.46

$$\begin{array}{|l} \mathsf{Cube}(a) \lor (\mathsf{Cube}(b) \rightarrow \mathsf{Tet}(c)) \\ \mathsf{Tet}(c) \rightarrow \mathsf{Small}(c) \\ (\mathsf{Cube}(b) \rightarrow \mathsf{Small}(c)) \rightarrow \mathsf{Small}(b) \\ \hline \neg\mathsf{Cube}(a) \rightarrow \mathsf{Small}(b) \end{array}$$

8.47

$$\begin{array}{|l} \mathsf{Small}(a) \land (\mathsf{Medium}(b) \lor \mathsf{Large}(c)) \\ \mathsf{Medium}(b) \rightarrow \mathsf{FrontOf}(a,b) \\ \mathsf{Large}(c) \rightarrow \mathsf{Tet}(c) \\ \hline \neg\mathsf{Tet}(c) \rightarrow \mathsf{FrontOf}(c,b) \end{array}$$

8.48

$$\begin{array}{|l} \mathsf{Small}(a) \land (\mathsf{Medium}(b) \lor \mathsf{Large}(c)) \\ \mathsf{Medium}(b) \rightarrow \mathsf{FrontOf}(a,b) \\ \mathsf{Large}(c) \rightarrow \mathsf{Tet}(c) \\ \hline \neg\mathsf{Tet}(c) \rightarrow \mathsf{FrontOf}(a,b) \end{array}$$

8.49

$$\begin{array}{|l} (\mathsf{Dodec}(a) \land \mathsf{Dodec}(b)) \\ \qquad \rightarrow (\mathsf{SameCol}(a,c) \rightarrow \mathsf{Small}(a)) \\ (\neg\mathsf{SameCol}(b,c) \land \neg\mathsf{Small}(b)) \\ \qquad \rightarrow (\mathsf{Dodec}(b) \land \neg\mathsf{Small}(a)) \\ \mathsf{SameCol}(a,c) \land \neg\mathsf{SameCol}(b,c) \\ \hline \mathsf{Dodec}(a) \rightarrow \mathsf{Small}(b) \end{array}$$

8.50

$$\begin{array}{|l} \mathsf{Cube(b)} \leftrightarrow (\mathsf{Cube(a)} \leftrightarrow \mathsf{Cube(c)}) \\ \hline \mathsf{Dodec(b)} \rightarrow (\mathsf{Cube(a)} \leftrightarrow \neg\mathsf{Cube(c)}) \end{array}$$

8.51

$$\begin{array}{|l} \mathsf{Cube(b)} \leftrightarrow (\mathsf{Cube(a)} \leftrightarrow \mathsf{Cube(c)}) \\ \hline \mathsf{Dodec(b)} \rightarrow \mathsf{a} \neq \mathsf{b} \end{array}$$

8.52

$$\begin{array}{|l} \mathsf{Cube(b)} \leftrightarrow (\mathsf{Cube(a)} \leftrightarrow \mathsf{Cube(c)}) \\ \hline \mathsf{Dodec(b)} \rightarrow \mathsf{a} \neq \mathsf{c} \end{array}$$

8.53

$$\begin{array}{|l} \mathsf{Small(a)} \rightarrow \mathsf{Small(b)} \\ \mathsf{Small(b)} \rightarrow (\mathsf{SameSize(b,c)} \rightarrow \mathsf{Small(c)}) \\ \neg\mathsf{Small(a)} \rightarrow (\mathsf{Large(a)} \land \mathsf{Large(c)}) \\ \hline \mathsf{SameSize(b,c)} \rightarrow (\mathsf{Large(c)} \lor \mathsf{Small(c)}) \end{array}$$